SUPPLEMENTARY MATERIALS: ANALYSIS AND CALIBRATION OF A LINEAR MODEL FOR STRUCTURED CELL POPULATIONS WITH UNIDIRECTIONAL MOTION : APPLICATION TO THE MORPHOGENESIS OF OVARIAN FOLLICLES*

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SM1. Supplemental proofs.

SM1.1. Deterministic model.

Proof of corollary 2.10. According to hypothesis 2.9,

$$\exists A > 0,\, \epsilon > 0 \text{ such that } \forall a \geq A,\, b_j(a) + \lambda_j > \epsilon. \tag{SM1}$$

Let $k \in \mathbb{N}$. Using hypothesis 2.2, for all $t \geq A$, we have:

$$0 \leq t^k b_j(t) e^{-\int_0^t [b_j(s)+\lambda_j]\, ds} \leq \bar{b}_j t^k e^{-\int_0^A [b_j(s)+\lambda_j]\, ds} e^{-\int_A^t [b_j(s)+\lambda_j]\, ds}.$$

Then, using (SM1) we obtain:

$$0 \leq t^k b_j(t) e^{-\int_0^t [b_j(s)+\lambda_j]\, ds} \leq t^k K_{A,\epsilon} e^{-\epsilon t},$$

where $K_{A,\epsilon}$ is a constant given by $K_{A,\epsilon} := \bar{b}_j e^{-\int_0^A [b_j(s)+\lambda_j-\epsilon]\, ds}$. As $\epsilon > 0$, the function $t \mapsto t^k e^{-\epsilon t}$ is integrable on $\mathbb{R}_+$, and we deduce that $\int_A^{+\infty} t^k e^{-\lambda_j t} d\mathcal{B}_j(t) dt < \infty$. Using the continuity of b_j (hypothesis 2.2), we conclude that $t \mapsto e^{-\lambda_j t} d\mathcal{B}_j(t) dt$ is integrable on $\mathbb{R}_+$. ∎

Proof of corollary 3.1. According to (16), we obtain:

$$\forall j \in [\![1, \lambda_c]\!],\, \frac{\phi^{(j)}(a)}{2[p_S^{(j)}\phi^{(j)}(0) + p_L^{(j)}\phi^{(j+1)}(0)]} = \int_a^{+\infty} b_j(s) e^{-\int_a^s \lambda_c + b_j(u) du} ds\,. \tag{SM2}$$

According to remark 2.5 and hypothesis 2.7, we deduce that $\lambda_c > -\bar{b}_j$, $\forall j \in [\![1, J]\!]$. Hence, using also hypothesis 2.2, we have:

$$\frac{\phi^{(j)}(a)}{2[p_S^{(j)}\phi^{(j)}(0) + p_L^{(j)}\phi^{(j+1)}(0)]} \geq \underline{b}_j \int_a^{+\infty} e^{-(\lambda_c + \bar{b}_j)(s-a)} ds = \frac{\underline{b}_j}{\lambda_c + \bar{b}_j},$$

and reminding that $\lambda_c > 0$ (see remark 2.5), we also obtain the right-side of (17):

$$\begin{aligned}\frac{\phi^{(j)}(a)}{2[p_S^{(j)}\phi^{(j)}(0) + p_L^{(j)}\phi^{(j+1)}(0)]} &\leq \int_a^{+\infty} b_j(s) e^{-\int_a^s b_j(u) du} ds \\ &= [-e^{-\int_a^s b_j(u) du}]_a^{+\infty} = 1\,.\end{aligned}$$ ∎

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Proof of lemma 3.2. For $j \in [\![1, J]\!]$, any solution of (4) in $\mathbf{L}^1(\mathbb{R}_+)$ is given by:

$$\hat{\phi}^{(j)}(a) = \hat{\phi}^{(j)}(0) e^{\int_0^a [\lambda_j + b_j(s)] ds} [1 - 2p_S^{(j)} \int_0^a b_j(s) e^{-\int_0^s [\lambda_j + b_j(u)] du} ds] .$$

According to hypothesis 2.4, $1 = 2p_S^{(j)} \int_0^{+\infty} b_j(s) e^{-\int_0^s [\lambda_j + b_j(u)] du} ds$, thus

$$\hat{\phi}^{(j)}(a) = 2p_S^{(j)} \hat{\phi}^{(j)}(0) \int_a^{+\infty} b_j(s) e^{-\int_a^s [\lambda_j + b_j(u)] du} ds .$$

Finally, according to remark 2.5, $\lambda_j > -\overline{b}_j$ and we obtain, using hypothesis 2.2,

$$\begin{aligned} \frac{\hat{\phi}^{(j)}(a)}{\hat{\phi}^{(j)}(0)} = 2p_S^{(j)} \int_a^{+\infty} b_j(s) e^{-\int_a^s [\lambda_j + b_j(u)] du} ds & \\ \geq \quad 2p_S^{(j)} \underline{b}_j \int_a^{+\infty} e^{-(\lambda_j + \overline{b}_j)(s-a)} ds = 2p_S^{(j)} \frac{\underline{b}_j}{\lambda_j + \overline{b}_j} . \end{aligned}$$

Then, we want to show that $\hat{\phi}^{(j)}(a) < \infty$ for all $a \in \mathbb{R}_+ \cup \{\infty\}$. Let

$$I(a) := \int_a^{+\infty} b_j(s) e^{-\int_0^s [\lambda_j + b_j(u)] du} ds .$$

Applying an integration by part to $I(a)$, we obtain that, for all $a \geq 0$,

$$I(a) = \left[e^{-\int_0^s [\lambda_j + b_j(u)] du} \right]_a^{\infty} - \lambda_j \int_a^{\infty} e^{-\int_0^s [\lambda_j + b_j(u)] du} ds .$$

Hypotheses 2.4 and 2.2 imply that, for all $a \geq 0$, $\int_a^{\infty} e^{-\int_0^s [\lambda_j + b_j(s)] ds} < \infty$ and so, $\lim_{s \to 0} e^{-\int_0^s [\lambda_j + b_j(u)] du} = 0$. Thus, we have:

$$\text{(SM3)} \qquad I(a) = e^{-\int_0^a [\lambda_j + b_j(u)] du} - \lambda_j \int_a^{\infty} e^{-\int_0^s [\lambda_j + b_j(u)] du} ds .$$

Multiplying (SM3) by $e^{\int_0^a [\lambda_j + b_j(u)] du}$, we deduce:

$$\text{(SM4)} \qquad \frac{\hat{\phi}^{(j)}(a)}{2p_S^{(j)} \hat{\phi}^{(j)}(0)} = 1 - \lambda_j \int_a^{\infty} e^{-\int_a^s [\lambda_j + b_j(u)] du} ds .$$

If $\lambda_j \geq 0$, we deduce directly from (SM4) that, for all $a \in \mathbb{R}_+ \cup \{\infty\}$, $\frac{\hat{\phi}^{(j)}(a)}{2p_S^{(j)} \hat{\phi}^{(j)}(0)} \leq 1$. We assume that $\lambda_j < 0$. Using hypothesis 2.9, we deduce that there exists constants $A > 0$ and $\epsilon > 0$ such that

$$\forall a \geq A, \quad \lambda_j + b_j(a) > \epsilon > 0.$$

Hence, with $C = \frac{-\lambda_j}{\epsilon} > 0$, we have:

$$\forall a \geq A, \quad -\lambda_j \leq C(\lambda_j + b_j(a)) .$$

Applying this inequality to (SM4), we obtain:

$$\forall a \geq A, \quad \frac{\hat{\phi}^{(j)}(a)}{2p_S^{(j)} \hat{\phi}^{(j)}(0)} \leq 1 + C \int_a^{\infty} [\lambda_j + b_j(s)] e^{-\int_0^s [\lambda_j + b_j(u)] du} ds \times e^{\int_0^a [\lambda_j + b_j(s)] ds} .$$

Again, using hypotheses 2.4 and 2.2, we obtain:

$$\int_a^{\infty} [\lambda_j + b_j(s)] e^{-\int_0^s [\lambda_j + b_j(u)]du} ds = \left[-e^{-\int_0^s [\lambda_j + b_j(u)]du} \right]_a^{\infty} = e^{-\int_0^a [\lambda_j + b_j(u)]du} .$$

We deduce

$$\forall a \geq A, \quad \frac{\hat{\phi}^{(j)}(a)}{2 p_S^{(j)} \hat{\phi}^{(j)}(0)} \leq 1 + C .$$

As $\hat{\phi}^{(j)}$ is continuous, we conclude that

$$\forall a \in \mathbb{R}_+ \cup \{+\infty\}, \quad \frac{\hat{\phi}^{(j)}(a)}{2 p_S^{(j)} \hat{\phi}^{(j)}(0)} < \infty .$$

∎

Proof of lemma 3.3. Deriving $\ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg$ with respect to t, we obtain

$$\frac{d}{dt} \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg = -e^{-\lambda_c t} \ll (\lambda_c \mathbb{1} + \mathcal{B} + \partial_a) \rho(t,\cdot), \phi \gg .$$

By integration by part and using that $\rho \in \mathbf{L}^1(\mathbb{R}_+)^J$ and $\phi \in \mathcal{C}_b^1(\mathbb{R}_+)^J$, we have

$$\ll \partial_a \rho(t,\cdot), \phi \gg = -\rho(t,0)^T \phi(0) - \ll \rho(t,\cdot), \partial_a \phi \gg ,$$

and we deduce

$$\frac{d}{dt} \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg = e^{-\lambda_c t} \left[\rho(t,0)^T \phi(0) + \ll \rho(t,\cdot), \partial_a \phi \gg \right.$$
$$\left. - \ll (\lambda_c \mathbb{1} + \mathcal{B}) \rho(t,\cdot), \phi \gg \right] .$$

As we have $(\lambda_c \mathbb{1} + \mathcal{B})^T = (\lambda_c \mathbb{1} + \mathcal{B})$, it comes $\ll \rho(t,\cdot), \partial_a \phi \gg - \ll (\lambda_c \mathbb{1} + \mathcal{B}) \rho(t,\cdot), \phi \gg = \ll \rho(t,\cdot), \partial_a \phi - (\lambda_c \mathbb{1} + \mathcal{B}) \phi \gg$. Then, using that $\mathcal{L}^D \phi = \lambda_c \phi$, we deduce $(\partial_a - \lambda_c \mathbb{1} - \mathcal{B}) \phi = -K(\cdot)^T \phi(0)$. Thus,

$$\frac{d}{dt} \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg = e^{-\lambda_c t} \left[\rho(t,0)^T \phi(0) - \ll \rho(t,\cdot), K(\cdot)^T \phi(0) \gg \right] .$$

Note that $\ll \rho(t,\cdot), K(\cdot)^T \phi(0) \gg = \ll K(\cdot) \rho(t,\cdot), \phi(0) \gg = \rho(t,0)^T \phi(0)$. Consequently,

$$\frac{d}{dt} \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg = 0 .$$

Hence,

$$\forall t, \quad \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg \quad = \quad \ll \rho_0(\cdot), \phi \gg \quad = \eta .$$

Thanks to the renormalization $\ll \hat{\rho}, \phi \gg = 1$, we obtain the conservation principle:

$$\ll e^{-\lambda_c t} \rho(t,\cdot) - \eta \hat{\rho}, \phi \gg = \ll e^{-\lambda_c t} \rho(t,\cdot), \phi \gg - \eta \ll \hat{\rho}, \phi \gg = 0 .$$

∎

Proof of lemma 3.4. From the linearity of the system, it can be easily shown that h is solution of

$$\begin{cases} \partial_t h(t,a) + \partial_a h(t,a) + [\lambda_c + B(a)]h(t,a) = 0, \quad t \geq 0, \quad a \geq 0, \\ h(t, a=0) = \int_0^\infty K(a)h(t,a)da. \end{cases}$$

Let f be a derivable function. Applying the chain rules, it comes, for $j \in [\![1, J]\!]$,

$$\begin{aligned} \partial_t f[h^{(j)}(t,a)] + \partial_a f[h^{(j)}(t,a)] &= f'(h^{(j)}(t,a))[\partial_t h^{(j)}(t,a) + \partial_a h^{(j)}(t,a)] \\ &= -[\lambda_c + b_j(a)] \times h^{(j)}(t,a) f'(h^{(j)}(t,a)) \,. \end{aligned}$$

For $f(x) = |x|$, $f'(x) = \frac{|x|}{x}$, we deduce

$$\partial_t |h^{(j)}(t,a)| + \partial_a |h^{(j)}(t,a)| = -[\lambda_c + b_j(a)]|h^{(j)}(t,a)| \,.$$

□

LEMMA SM1.1. *[Modified Grönwall lemma] Let $N \in \mathbb{N}^*$. Suppose that $\forall i \in [\![1, N]\!]$, there exist $\kappa_i \in \mathbb{R}_+^*$, $\gamma \in \mathbb{R}_+^*$ and P_i polynomials of degree $\alpha_i \in \mathbb{N}$ such that*

$$F'(t) \leq \sum_{i=1}^{N} P_i(t) e^{-\kappa_i t} - \gamma F(t).$$

Then,

$$F(t) \leq K e^{-\gamma t} + \sum_{i=1}^{N} \widetilde{P}_i(t) e^{-\kappa_i t},$$

where K is a constant and for all $i \in [\![1, N]\!]$, $\widetilde{P}_i$ is a polynomial of degree $\widetilde{\alpha}_i \leq \alpha_i + 1$.

Proof. Note that $\frac{d}{dt}(e^{\gamma t} F(t)) = (F'(t) + \gamma F(t)) \times e^{\gamma t}$. Hence,

$$\frac{d}{dt}(e^{\gamma t} F(t)) \leq \sum_{i=1}^{N} P_i(t) e^{(-\kappa_i + \gamma)t}.$$

Then, integrating on the interval $[0, t]$, we obtain:

$$e^{\gamma t} F(t) - F(0) \leq \sum_{i=1}^{N} \widetilde{P}_i(t) e^{(\gamma - \kappa_i)t} + K$$

$$F(t) \leq (F(0) + K) e^{-\gamma t} + \sum_{i=1}^{N} \widetilde{P}_i(t) e^{-\kappa_i t},$$

where K is a constant and for all $i \in [\![1, N]\!]$, $\widetilde{P}_i$ a polynomial of degree $\widetilde{\alpha}_i \leq \alpha_i + 1$ (the degree increases when $\gamma = \kappa_i$). □

SM1.2. Stochastic model. For any $f : (t,a) \mapsto (f_t^{(j)}(a))_{j \in [\![1,J]\!]} \in \mathcal{B}_b^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})^J$ (the space product of the set of bounded functions with bounded derivatives), we note ∂_1 and ∂_2 respectively its derivative with respect to time (t) and age (a).

LEMMA SM1.2. *Let $F \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, $f \in \mathcal{B}^1_b(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})^J$.*

$$
\begin{aligned}
F[\ll Z_t, f_t \gg] = F[\ll f_0, Z_0 \gg] &+ \int_0^t \ll \partial_1 f_s + \partial_2 f_s, Z_s \gg F'[\ll f_s, Z_s \gg] ds \\
&+ \int_{[0,t]\times\mathcal{E}} \Big[\mathbb{1}_{k\le N_{s^-}} \Big(F[\ll f_s, 2\delta_{I^{(k)}_{s^-},0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} + Z_{s^-} \gg] \\
&\qquad - F[\ll f_s, Z_{s^-} \gg] \Big) \mathbb{1}_{0\le\theta\le m_1(s,k,Z)} \\
&+ \Big(F[\ll f_s, \delta_{I^{(k)}_{s^-}+1,0} + \delta_{I^{(k)}_s,0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} + Z_{s^-} \gg] \\
&\qquad - F[\ll f_s, Z_{s^-} \gg] \Big) \mathbb{1}_{m_1(s,k,Z)\le\theta\le m_2(s,k,Z)} \\
&+ \Big(F[\ll f_s, 2\delta_{I^{(k)}_{s^-}+1,0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} + Z_{s^-} \gg] \\
&\qquad - F[\ll f_s, Z_{s^-} \gg] \Big) \mathbb{1}_{m_2(s,k,Z)\le\theta\le m_3(s,k,Z)} \Big] Q(ds,dk,d\theta).
\end{aligned}
$$

Proof. We integrate f_t against the measure Z_t

$$
\begin{aligned}
&\ll Z_t, f_t \gg = \sum_{k=1}^{N_0} f_t^{(I_0^{(k)})}(A_0^{(k)} + t) \\
&+ \int_{[0,t]\times\mathcal{E}} \Big[\mathbb{1}_{k\le N_{s^-}} \Big(2 f_t^{(I^{(k)}_{s^-})}(t-s) - f_t^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-} + t - s) \Big) \mathbb{1}_{0\le\theta\le m_1(s,k,Z)} \\
&+ \Big(f_t^{(I^{(k)}_{s^-})}(t-s) + f_t^{(I^{(k)}_{s^-}+1)}(t-s) - f_t^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-} + t - s) \Big) \mathbb{1}_{m_1(s,k,Z)\le\theta\le m_2(s,k,Z)} \\
&+ \Big(2 f_t^{(I^{(k)}_{s^-}+1)}(t-s) - f_t^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-} + t - s) \Big) \mathbb{1}_{m_2(s,k,Z)\le\theta\le m_3(s,k,Z)} \Big] Q(ds,dk,d\theta).
\end{aligned}
$$

Derivating $f_t^{(j)}[a+t-s]$, we obtain

$$
\begin{aligned}
&\tfrac{d}{dt}\Big[f_t^{(j)}[a+t-s] \Big] = \partial_1 f_t^{(j)}[a+t-s] + \partial_2 f_t^{(j)}[a+t-s] \\
&\Rightarrow \quad \int_s^t \tfrac{d}{du}\Big[f_u^{(j)}[a+u-s] \Big] du = \int_s^t \Big[\partial_1 f_u^{(j)}[a+u-s] + \partial_2 f_u^{(j)}[a+u-s] \Big] du \\
&\Rightarrow f_t^{(j)}[a+t-s] = f_s^{(j)}[a] + \int_s^t \Big[\partial_1 f_u^{(j)}[a+u-s] + \partial_2 f_u^{(j)}[a+u-s] \Big] du.
\end{aligned}
$$

Then, replacing j by the index $I^{(k)}_{s^-}$ and a by $A^{(k)}_{s^-}$ or 0, it comes

$$
\begin{aligned}
&\ll Z_t, f_t \gg = \textstyle\sum_{k=1}^{N_0} f_0^{(I_0^{(k)})}(A_0^{(k)}) + T_0 + T_1 + T_2 + T_3 \\
&\quad + \textstyle\int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k\le N_{s^-}} \Big[\Big(2 f_s^{(I^{(k)}_{s^-})}(0) - f_s^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-}) \Big) \mathbb{1}_{0\le\theta\le m_1(s,k,Z)} \\
&\quad + \Big(f_s^{(I^{(k)}_{s^-})}(0) + f_s^{(I^{(k)}_{s^-}+1)}(0) - f_s^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-}) \Big) \mathbb{1}_{m_1(s,k,Z)\le\theta\le m_2(s,k,Z)} \\
&\quad + \Big(2 f_s^{(I^{(k)}_{s^-}+1)}(0) - f_s^{(I^{(k)}_{s^-})}(A^{(k)}_{s^-}) \Big) \mathbb{1}_{m_2(s,k,Z)\le\theta\le m_3(s,k,Z)} \Big] Q(ds,dk,d\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
T_0 = &\sum_{k=1}^{N_0} \int_0^t \left[\partial_1 f_u^{(I_0^{(k)})}(A_0^{(k)} + u) + \partial_2 f_u^{(I_0^{(k)})}(A_0^{(k)} + u) \right] du, \\
T_1 = &\int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \int_s^t \Big[2\partial_1 f_u^{(I_{s^-}^{(k)})}(u-s) \\
&+ 2\partial_2 f_u^{(I_{s^-}^{(k)})}(u-s) - \partial_1 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \\
&- \partial_2 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \mathbb{1}_{0 \le \theta \le m_1(s,k,Z)} \Big] du Q(ds, dk, d\theta), \\
T_2 = &\int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \int_s^t \Big[\partial_1 f_u^{(I_{s^-}^{(k)})}(u-s) + \partial_1 f_u^{(I_{s^-}^{(k)}+1)}(u-s) + \partial_2 f_u^{(I_{s^-}^{(k)})}(u-s) \\
&+ \partial_2 f_u^{(I_{s^-}^{(k)}+1)}(u-s) - \partial_1 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \\
&- \partial_2 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \mathbb{1}_{m_1(s,k,Z) \le \theta \le m_2(s,k,Z)} \Big] du Q(ds, dk, d\theta), \\
T_3 = &\int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \int_s^t \Big[2\partial_1 f_u^{(I_{s^-}^{(k)}+1)}(u-s) \\
&+ 2\partial_2 f_u^{(I_{s^-}^{(k)}+1)}(u-s) - \partial_1 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \\
&- \partial_2 f_u^{(I_{s^-}^{(k)})}(A_{s^-}^{(k)} + u - s) \mathbb{1}_{m_2(s,k,Z) \le \theta \le m_3(s,k,Z)} \Big] du Q(ds, dk, d\theta) \,.
\end{aligned}
$$

As the partial differential of each $f^{(j)}$ are uniformly bounded, we can apply Fubini theorem on T_0, T_1, T_2 and T_3:

$$
\begin{aligned}
T_0 = &\int_0^t \ll \partial_1 f_u + \partial_2 f_u, \sum_{k=1}^{N_0} \delta_{I_0^{(k)}, A_0^{(k)} + u} \gg du, \\
T_1 = &\int_0^t \Big[\ll \partial_1 f_u + \partial_2 f_u, \int_0^u \int_{\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \Big(2\delta_{I_{s^-}^{(k)}, u-s} \\
&- \delta_{I_{s^-}^{(k)}, A_{s^-}^{(k)} + u - s} \Big) \mathbb{1}_{0 \le \theta \le m_1(s,k,Z)} Q(ds, dk, d\theta) \gg \Big] du, \\
T_2 = &\int_0^t \Big[\ll \partial_1 f_u + \partial_2 f_u, \int_0^u \int_{\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \Big(\delta_{I_{s^-}^{(k)}, u-s} + \delta_{I_{s^-}^{(k)}+1, u-s} \\
&- \delta_{I_{s^-}^{(k)}, A_{s^-}^{(k)} + u - s} \Big) \mathbb{1}_{m_1(s,k,Z) \le \theta \le m_2(s,k,Z)} Q(ds, dk, d\theta) \gg \Big] du, \\
T_3 = &\int_0^t \Big[\ll \partial_1 f_u + \partial_2 f_u, \int_0^u \int_{\mathcal{E}} \mathbb{1}_{k \le N_{s^-}} \Big(2\delta_{I_{s^-}^{(k)}+1, u-s} \\
&- \delta_{I_{s^-}^{(k)}, A_{s^-}^{(k)} + u - s} \Big) \mathbb{1}_{m_2(s,k,Z) \le \theta \le m_3(s,k,Z)} Q(ds, dk, d\theta) \gg \Big] du.
\end{aligned}
$$

Finally, using the stochastic differential equation (2)

$$T_0 + T_1 + T_2 + T_3 = \int_0^t \ll \partial_1 f_u + \partial_2 f_u, Z_u \gg du.$$

Consequently, we obtain

$$\begin{aligned}
\ll f_t, Z_t \gg = \ll f_0, Z_0 \gg &+ \int_0^t \ll \partial_1 f_s + \partial_2 f_s, Z_s \gg ds \\
&+ \int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k\le N_{s^-}} \Big[\ll f_s, 2\delta_{I^{(k)}_{s^-},0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} \gg \mathbb{1}_{0\le\theta\le m_1(s,k,Z)} \\
&+ \ll f_s, \delta_{I^{(k)}_{s^-},0} + \delta_{I^{(k)}_{s^-}+1,0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} \gg \mathbb{1}_{m_1(s,k,Z)\le\theta\le m_2(s,k,Z)} \\
&+ \ll f_s, 2\delta_{I^{(k)}_{s^-}+1,0} - \delta_{I^{(k)}_{s^-},A^{(k)}_{s^-}} \gg \mathbb{1}_{m_2(s,k,Z)\le\theta\le m_3(s,k,Z)} \Big] Q(ds,dk,d\theta),
\end{aligned}$$

which gives us lemma SM1.2 for $F(x) = x$. We conclude by applying the Ito's formula (see [SM5], p.68-70). ∎

We introduce the sequence of stopping times ξ_N.

Definition SM1.3. *Let ξ_N a sequence of stopping times defined as*

$$\xi_N = \sup\big(t : N_t < N, \quad \ll a, Z_t \gg < N\big).$$

Proof of theorem 3.6. We first start by showing (32).

$$N_t = N_0 + \int_{[0,t]\times\mathcal{E}} \mathbb{1}_{k\le N_{s^-}} \mathbb{1}_{0\le\theta\le b_{I^{(k)}_{s^-}}(A^{(k)}_{s^-})} Q(ds,dk,d\theta).$$

Thus,

$$\sup_{s\le t\wedge\xi_N} N_s \le N_0 + \int_0^{t\wedge\xi_N} \int_{\mathcal{E}} \mathbb{1}_{k\le N_{s^-}} \mathbb{1}_{0\le\theta\le\bar{b}} Q(ds,dk,d\theta),$$

where $\bar{b} := \sup_{j\in[\![0,J]\!]} \bar{b}_j$. Taking the expectation and using Poisson measure properties, we obtain

$$\mathbb{E}[\sup_{s\le t\wedge\xi_N} N_s] \le \mathbb{E}[N_0] + \bar{b}\mathbb{E}\left[\int_0^{t\wedge\xi_N} N_s ds\right] \le \mathbb{E}[N_0] + \bar{b}\mathbb{E}\left[\int_0^{t\wedge\xi_N} \sup_{u\le s\wedge\xi_N} N_u ds\right].$$

Hence,

$$\mathbb{E}[\sup_{s\le t\wedge\xi_N} N_s] \le \mathbb{E}[N_0] + \bar{b}\mathbb{E}\left[\int_0^t \sup_{u\le s\wedge\xi_N} N_u ds\right].$$

By Fubini theorem, we deduce that

$$\mathbb{E}[\sup_{s\le t\wedge\xi_N} N_s] \le \mathbb{E}[N_0] + \bar{b}\int_0^t \mathbb{E}\left[\sup_{u\le s\wedge\xi_N} N_u\right] ds.$$

Applying Grönwall lemma, we deduce for all $t \leq T$ that

$$\mathbb{E}[\sup_{s\leq t\wedge\xi_N} N_s] \leq \mathbb{E}[N_0]e^{\bar{b}t}.$$

Hence,

$$\mathbb{E}[\sup_{t\leq T\wedge\xi_N} N_t] \leq \mathbb{E}[N_0]e^{\bar{b}T} \quad < \infty\,.$$

Using the same method, we also deduce that $\mathbb{E}\big[\sup_{t\leq T} \ll a, Z_t \gg\big] < \infty$.

Then, we use the same approach as [SM8] (Theorem 2.2.8) to compute the infinitesimal generator of Z_t, denoted by $\mathcal{G}$. By construction, $(Z_t)_{t\in\mathbb{R}_+}$ is a markovian process of $\mathbb{D}([0,T],\mathcal{M}_P([\![1,J]\!]\times\mathbb{R}_+))$. Let $f \in \mathbf{C}^1_b(\mathcal{E},\mathbb{R})$, by definition, $\mathcal{G}F := \lim_{t\to 0}\frac{d}{dt}\mathbb{E}\big[F[\ll f, Z_t \gg]\big]$. Taking the expectation of the expression of $\ll f, Z_{t\wedge\xi_N} \gg$ given in lemma SM1.2, we obtain

$$\begin{aligned}\mathbb{E}\left[F[\ll f, Z_{t\wedge\xi_N} \gg]\right] &= \mathbb{E}\left[F[\ll f, Z_0 \gg]\right] \\ &+\mathbb{E}\left[\int_0^{t\wedge\xi_N} \ll \partial_a f, Z_s \gg F'[\ll f, Z_s \gg]ds\right] + \mathbb{E}\left[\chi^{f,F}(t\wedge\xi_N, Z)\right],\end{aligned}$$

where

$$\begin{aligned}\chi^{f,F}(t,Z) := \quad & \int_0^t\int_{\mathcal{E}}\Big[\left(F[\ll f, 2\delta_{j,0}-\delta_{j,a}+Z_s \gg] - F[< f, Z_s >]\right)p^{(j)}_{2,0} \\ & +\left(F[< f, \delta_{j+1,0}+\delta_{j,0}-\delta_{j,a}+Z_s >] - F[\ll f, Z_s \gg]\right)p^{(j)}_{1,1} \\ & +\left(F[< f, 2\delta_{j+1,0}-\delta_{j,a}+Z_s >]\right. \\ & \left.-F[< f, Z_s >]\right)p^{(j)}_{0,2}\Big]\, b_j(a)Z_s(dj,da)ds.\end{aligned}$$

We have the following estimates,

$$\begin{aligned}&\mathbb{E}\left[\chi^{f,F}(t\wedge\xi_N, Z)\right] \leq 2\mathbb{E}\left[\sup_{t\leq T} N_t\right] T\|F\|_\infty \bar{b}, \\ &\mathbb{E}[\textstyle\int_0^{t\wedge\xi_N} \ll \partial_a f_s, Z_s \gg F'[\ll f, Z_s \gg]ds] \leq \mathbb{E}\big[\sup_{t\leq T} N_t\big]T\times\|\partial_a f\|_\infty\times\|F'\|_\infty.\end{aligned}$$

Those bounds are independent of N thanks to (32), so that we may let N goes to infinity. Moreover,

$$\frac{d}{dt}\int_0^t \ll \partial_a f, Z_s \gg F'[\ll f, Z_s \gg]ds = \ll \partial_a f, Z_t \gg F'[\ll f, Z_t \gg]$$

which is also dominated by $\mathbb{E}\big[\sup_{t\leq T} N_t\big]\times\|\partial_a f\|_\infty\times\|F'\|_\infty$. Also,

$$\begin{aligned}\tfrac{\partial}{\partial t}\chi(t,Z_t) = \textstyle\int_{\mathcal{E}}&\Big[\left(F[< f, 2\delta_{j,0}-\delta_{j,a}+Z_t >] - F[< f, Z_t >]\right)p^{(j)}_{2,0} \\ & +\left(F[\ll f, \delta_{j+1,0}+\delta_{j,0}-\delta_{j,a}+Z_t \gg] - F[\ll f, Z_t \gg]\right)p^{(j)}_{1,1} \\ & +\left(F[\ll f, 2\delta_{j+1,0}-\delta_{j,a}+Z_t \gg] - F[\ll f, Z_t \gg]\right)p^{(j)}_{0,2}\Big]\, b_j(a)Z_t(dj,da).\end{aligned}$$

$|\frac{\partial}{\partial t}\chi(t,Z_t)|$ is dominated $\mathbb{P}$-p.s by $2\mathbb{E}\big[\sup_{t\leq T} N_t\big]\|F\|_\infty\bar{b}$. We can thus apply the differentiating theorem under the integral sign $\mathbb{E}$ and conclude. ∎

Proof of lemma 3.7. Introducing the compensated Poisson measure $\tilde{Q}$, $\tilde{Q}(ds,dk,d\theta) := Q(ds,dk,d\theta) - dsdkd\theta$, we define the process:

$$
\begin{aligned}
M_t^{F,f} := \quad & \int\int_{[0,t\wedge\xi_N]\times\mathcal{E}} \mathbb{1}_{k<N_{s-}} \Big[\Big(F[\ll f, 2\delta_{I^k_{s-},0} - \delta_{I^k_{s-},A^k_{s-}} + Z_{s-} \gg] \\
& -F[\ll f, Z_{s-} \gg]\Big)\,\mathbb{1}_{0\le\theta\le m_1(s,k,Z)} \\
& + \Big(F[\ll f, \delta_{I^k_{s-}+1,0} + \delta_{I^k_{s-},0} - \delta_{I^k_{s-},A^k_{s-}} + Z_{s-} \gg] \\
& -F[\ll f, Z_{s-} \gg]\Big)\,\mathbb{1}_{m_1(s,k,Z)\le\theta\le m_2(s,k,Z)} \\
& + \Big(F[\ll f, 2\delta_{I^k_{s-}+1,0} - \delta_{I^k_{s-},A^k_{s-}} + Z_{s-} \gg] \\
& -F[\ll f, Z_{s-} \gg]\Big)\,\mathbb{1}_{m_2(s,k,Z)\le\theta\le m_3(s,k,Z)}\Big]\,\tilde{Q}(ds,dk,d\theta).
\end{aligned}
$$

We can verify that $M_t^{F,f}$ is a martingale as an integral against a compensated Poisson measure. Then, applying lemma SM1.2 and the definition of the generator given in theorem 3.6, we show that

$$
M_t^{F,f} = F[\ll f, Z_t \gg] - F[\ll f, Z_0 \gg] - \int_0^t \mathcal{G}F[\ll f, Z_s \gg]ds\,. \tag{SM5}
$$

We turn now to the computation of the quadratic variation and use the same approach as in [SM1]. We apply (SM5) for $F(x) = x^2$. Note that we cannot use directly this result as $x \mapsto x^2$ is not bounded and we need to first use a localizing sequence (see [SM4] p. 382, theorem 13.14). We obtain that

$$
\begin{aligned}
& \ll f, Z_t \gg^2 - \ll f, Z_0 \gg^2 - \int_0^t 2 \ll f, Z_s \gg \times \ll \partial_a f, Z_s \gg ds \\
& - \int_0^t \sum_{j=1}^J \int_{\mathbb{R}_+} \Big[\big(\ll f, 2\delta_{j,0} - \delta_{j,a} + Z_s \gg^2 - \ll f, Z_s \gg^2\big)\, b_j(a) p_{2,0}^{(j)} \\
& - \big(\ll f, \delta_{j,0} + \delta_{j+1,0} - \delta_{j,a} + Z_s \gg^2 - \ll f, Z_s \gg^2\big)\, b_j(a) p_{1,1}^{(j)} Z_s(dj,da) \\
& - \big(\ll f, 2\delta_{j+1,0} - \delta_{j,a} + Z_s \gg^2 - \ll f, Z_s \gg^2\big)\, b_j(a) p_{0,2}^{(j)} Z_s(dj,da) ds\Big]
\end{aligned}
\tag{SM6}
$$

is a martingale. Then, applying (SM5) for $F(x) = x$ (using a localizing sequence again), we get that

$$
\begin{aligned}
\ll f, Z_t \gg \quad = \quad & \ll f, Z_0 \gg + \int_0^t \ll \partial_a f, Z_s \gg ds \\
+ & \int_0^t \big[\sum_{j=1}^J \int_{\mathbb{R}_+} \ll f, 2\delta_{j,0} - \delta_{j,a} \gg b_j(a) p_{2,0}^{(j)} Z_s(dj,da)\big] ds \\
+ & \int_0^t \big[\sum_{j=1}^J \int_{\mathbb{R}_+} \ll f, \delta_{j,0} + \delta_{j+1,0} - \delta_{j,a} \gg b_j(a) p_{1,1}^{(j)} Z_s(dj,da)\big] ds \\
+ & \int_0^t \big[\sum_{j=1}^J \int_{\mathbb{R}_+} \ll f, 2\delta_{j+1,0} - \delta_{j,a} \gg b_j(a) p_{0,2}^{(j)} Z_s(dj,da)\big] ds + M_t^f
\end{aligned}
$$

is a semi-martingale. Applying the Ito formula (see [SM5], p. 78-79), we obtain:

$$\begin{aligned}
&\ll f, Z_t \gg^2 \quad - \quad \ll f, Z_0 \gg^2 - \int_0^t 2 \ll f, Z_s \gg \times \ll \partial_a f, Z_s \gg ds \\
&+ \sum_{j=1}^J \int_{\mathbb{R}_+} (2 \ll f, Z_s \gg \times \ll f, 2\delta_{j,0} - \delta_{j,a} \gg) \, b_j(a) p_{2,0}^{(j)} Z_s(dj, da) \\
&+ \sum_{j=1}^J \int_{\mathbb{R}_+} (2 \ll f, Z_s \gg \times \ll f, \delta_{j,0} + \delta_{j+1,0} - \delta_{j,a} \gg) \, b_j(a) p_{1,1}^{(j)} Z_s(dj, da) \\
&+ \sum_{j=1}^J \int_{\mathbb{R}_+} (2 \ll f, Z_s \gg \times \ll f, 2\delta_{j+1,0} - \delta_{j,a} \gg) \, b_j(a) p_{0,2}^{(j)} Z_s(dj, da) \big] ds \\
&- \left\langle M^f, M^f \right\rangle_t
\end{aligned} \tag{SM7}$$

is a martingale. We consider the jump corresponding to the case when the two daughter cells remain on their mother layer. Note that

$$\begin{aligned}
&\ll f, 2\delta_{j,0} - \delta_{j,a} + Z_s \gg^2 - \ll f, Z_s \gg^2 = \\
&\quad 2 \ll f, Z_s \gg \times \ll f, 2\delta_{j,0} - \delta_{j,a} \gg + \ll f, 2\delta_{j,0} - \delta_{j,a} \gg^2 .
\end{aligned}$$

We proceed similarly for the two other jumps. Applying the Doob-Meyer theorem ([SM5], p. 106), we deduce the quadratic variation $\left\langle M^f, M^f \right\rangle_t$ comparing (SM6) and (SM7). ∎

SM1.3. Moment study.

Generating functions.

Proof of lemma 3.9. Let $a \geq 0$. Remind that the generating function is given by

$$F^{(i,a)}[\mathbf{s}; t] = \sum_{\mathbf{k} \in \mathbb{N}^J} \mathbf{s}^{\mathbf{k}} \mathbb{P}\big[Y_t^a = \mathbf{k} | Z_0 = \delta_{i,0}\big].$$

Let $i \in [\![1, J]\!]$ and $\mathbf{j}, \mathbf{k} \in \mathbb{N}^J$. We note $P_{\mathbf{j},\mathbf{k}}^a(t) := \mathbb{P}\big[Y_t^a = \mathbf{k} | Z_0 = \sum_{i=1}^J j_i \delta_{i,0}\big]$. We write the backward equation for the probability $P_{e_i,\mathbf{k}}^a(t) := \mathbb{P}\big[Y_t^a = \mathbf{k} | Z_0 = \delta_{i,0}\big]$. Starting from a single mother cell of age 0 and layer i, there are three possibilities at time t: (i) the cell has not divided and $t \leq a$, (ii) the cell has not divided and $t > a$, and (iii) the cell has divided. Thus,

$$\begin{aligned}
P_{e_i,\mathbf{k}}^a(t) &= (\delta_{e_i,\mathbf{k}} \mathbb{1}_{t \leq a} + \delta_{\mathbf{0},\mathbf{k}} \mathbb{1}_{t > a}) \, \mathbb{P}[\tau^{(i)}(a_0 = 0) \geq t] \\
&+ \int_0^t [p_{2,0}^{(i)} P_{2e_i,\mathbf{k}}^a(t-y) + p_{1,1}^{(i)} P_{e_i+e_{i+1},\mathbf{k}}^a(t-y) + p_{0,2}^{(i)} P_{2e_{i+1},\mathbf{k}}^a(t-y)] d\mathcal{B}_i(y) dy
\end{aligned} \tag{SM8}$$

where $\mathbb{P}[\tau^{(i)}(a_0 = 0) \geq t] = e^{-\int_0^t b_i(s) ds} \mathbb{1}_{t \geq 0} = 1 - \mathcal{B}_i(t)$.

Applying the branching property, we have for all $y \in [0, t]$, for all $i \in [\![1, J]\!]$

$$P_{2e_i,\mathbf{k}}^a(y) = \sum_{\mathbf{k}_1, \mathbf{k}_2 / \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} [P_{e_i,\mathbf{k}_1}^a(y) P_{e_i,\mathbf{k}_2}^a(y)],$$

and also, for all $i \in [\![1, J-1]\!]$,

$$P_{e_i+e_{i+1},\mathbf{k}}^a(y) = \sum_{\mathbf{k}_1, \mathbf{k}_2 / \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}} P_{e_{i+1},\mathbf{k}_1}^a(y) P_{e_i,\mathbf{k}_2}^a(y) \, .$$

Hence, we can rewrite the expression of

$$A_t := \int_0^t [p_{2,0}^{(i)} P^a_{2e_i,\mathbf{k}}(t-y) + p_{1,1}^{(i)} P^a_{e_i+e_{i+1},\mathbf{k}}(t-y) + p_{0,2}^{(i)} P^a_{2e_{i+1},\mathbf{k}}(t-y)] d\mathcal{B}_i(y)dy$$

as

$$\begin{aligned}
A_t = p_{2,0}^{(i)} &\int_0^t \sum_{\mathbf{k_1},\mathbf{k_2}/\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} P^a_{e_i,\mathbf{k_1}}(t-y) P^a_{e_i,\mathbf{k_2}}(t-y) d\mathcal{B}_i(y)dy \\
&+ p_{1,1}^{(i)} \int_0^t \sum_{\mathbf{k_1},\mathbf{k_2}/\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} P^a_{e_{i+1},\mathbf{k_1}}(t-y) P^a_{e_i,\mathbf{k_2}}(t-y) d\mathcal{B}_i(y)dy \\
&+ p_{0,2}^{(i)} \int_0^t \sum_{\mathbf{k_1},\mathbf{k_2}/\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} P^a_{e_{i+1},\mathbf{k_1}}(t-y) P^a_{e_{i+1},\mathbf{k_2}}(t-y) d\mathcal{B}_i(y)dy.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k}} P^a_{2e_i,\mathbf{k}}(t-y) &= \sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k}} \sum_{\mathbf{k_1},\mathbf{k_2}/\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} P^a_{e_i,\mathbf{k_1}}(t-y) P^a_{e_i,\mathbf{k_2}}(t-y) \\
&= \sum_{\mathbf{k}\in\mathbb{N}^J} \sum_{\mathbf{k_1}=0}^{\mathbf{k}} \mathbf{s}^{\mathbf{k_1}} P^a_{e_i,\mathbf{k_1}}(t-y) \mathbf{s}^{\mathbf{k}-\mathbf{k_1}} P^a_{e_i,\mathbf{k}-\mathbf{k_1}}(t-y).
\end{aligned}$$

We note $\sum_{\mathbf{k_1}=0}^{\mathbf{k}}$ the sum of all the $\mathbf{k}_1 \in \mathbb{N}^J$ vectors such that $\mathbf{k}_1 \leq \mathbf{k}$ component by component. We have

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k}} P^a_{2e_i,\mathbf{k}}(t-y) &= \sum_{\mathbf{k}\in\mathbb{N}^J} \sum_{\mathbf{k_1}=0}^{\mathbf{k}} \mathbf{s}^{\mathbf{k_1}} P^a_{e_i,\mathbf{k_1}}(t-y) \mathbf{s}^{\mathbf{k}-\mathbf{k_1}} P^a_{e_i,\mathbf{k}-\mathbf{k_1}}(t-y) \\
&= \sum_{\mathbf{k_1}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k_1}} P^a_{e_i,\mathbf{k_1}}(t-y) \sum_{\mathbf{k}\geq\mathbf{k_1}} \mathbf{s}^{\mathbf{k}-\mathbf{k_1}} P^a_{e_i,\mathbf{k}-\mathbf{k_1}}(t-y) \\
&= (\sum_{\mathbf{k_1}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k_1}} P^a_{e_i,\mathbf{k_1}}(t-y))(\sum_{\mathbf{k_2}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k_2}} P^a_{e_i,\mathbf{k_2}}(t-y)).
\end{aligned}$$

Hence,

$$\sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k}} P^a_{2e_i,\mathbf{k}}(t-y) = (F^{(i,a)}[\mathbf{s}; t-y])^2. \tag{SM9}$$

In the same way, we also obtain

$$\sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{k} P^a_{e_i+e_{i+1},k}(t-y) = F^{(i,a)}[\mathbf{s}; t-y] F^{(i+1,a)}[\mathbf{s}; t-y] \tag{SM10}$$

and

$$\sum_{\mathbf{k}\in\mathbb{N}^J} \mathbf{s}^{\mathbf{k}} P^a_{2e_{i+1},\mathbf{k}}(t-y) = (F^{(i+1,a)}[\mathbf{s}; t-y])^2. \tag{SM11}$$

Finally, multiplying (SM8) by $\mathbf{s}^{\mathbf{k}}$, summing on $\mathbf{k} \in \mathbb{N}^J$ and applying (SM9)-(SM11), we obtain:

$$\forall i \in [\![1,J]\!], F^{(i,a)}[\mathbf{s};t] = (s_i \mathbb{1}_{t\leq a} + \mathbb{1}_{t>a})(1-\mathcal{B}_i(t)) + \int_0^t f^{(i)}(F[\mathbf{s};t-y])d\mathcal{B}_i(y)dy.$$

□

First moments.

Proof of lemma 3.10. By classical property, $M^a_{i,j}(t) = \frac{\partial}{\partial s_j} F^{(i,a)}[s;t]|_{\mathbf{s}=1}$. From (38) it comes that

$$\text{(SM12)} \quad \frac{\partial}{\partial s_j} F^{(i,a)}\big[s;t\big] = \delta_{i,j}(1-\mathcal{B}_{i,i}(t))\mathbb{1}_{a\geq t} + \int_0^t \frac{\partial}{\partial s_j} f^{(i)}\big[F^a(s,y)\big]d\mathcal{B}_i(t-y)dy$$

where

$$\frac{\partial}{\partial s_j} f^{(i)}[F^a(s,t)] = 2p^{(i)}_{2,0}F^{(i,a)}[s;t]\frac{\partial}{\partial s_j}F^{(i,a)}[s;t] + 2p^{(i)}_{0,2}F^{(i+1,a)}[s;t]\frac{\partial}{\partial s_j}F^{(i+1,a)}[s;t]$$
$$+ p^{(i)}_{1,1}\big[F^{(i+1,a)}[s;t]\frac{\partial}{\partial s_j}F^{(i,a)}[s;t] + F^{(i,a)}[s;t]\frac{\partial}{\partial s_j}F^{(i+1,a)}[s;t]\big].$$

For $\mathbf{s}=1$, knowing that $F^{(i,a)}(1,t)=1$, we get

$$M^a_{i,j}(t) = \delta_{i,j}(1-\mathcal{B}_i(t))\mathbb{1}_{t\leq a}$$
$$+ \int_0^t \big[2p^{(i)}_{2,0}M^a_{i,j}(y) + p^{(i)}_{1,1}[M^a_{i,j}(y) + M^a_{i+1,j}(y)] + 2p^{(i)}_{0,2}M^a_{i+1,j}(y)\big]d\mathcal{B}_i(t-y))dy$$

which can be rewritten as

$$M^a_{i,j}(t) = \delta_{i,j}(1-\mathcal{B}_i(t))\mathbb{1}_{t\leq a} + \big[2p^{(i)}_S M^a_{i,j} + 2p^{(i)}_L M^a_{i+1,j}\big] * d\mathcal{B}_i(t).$$

□

Harris lemmas. We recall some results on the renewal theory presented in [SM3], p.161-163.
Let G be a distribution function on $(0,\infty)$ with the additional assumption $G(0+)=0$.
We consider the renewal equation

$$\text{(SM13)} \qquad K(t) = f(t) + m\int_0^t K(t-u)dG(u) = f(t) + mK*G(t)$$

where m is a positive constant representing the mean number of children, f is a continuous function representing a source term and G is the life time distribution. In addition, we suppose that G is not lattice.

Lemma SM1.4 (Harris's lemma 2, p.161). *Suppose that there exists a Malthus parameter α such that $m\int_0^\infty e^{-\alpha t}dG(t) = 1$, and that the following conditions also hold:*

(a) $f(t)e^{-\alpha t}$ *is a continuous function such that* $f(t)e^{-\alpha t} \in \mathbf{L}^1(\mathbb{R}_+)$.

(b) $\int_0^\infty t^2 dG(t) < \infty$.

Then, $K(t) \sim n_f e^{\alpha t}$, *where*

$$n_f = \frac{\int_0^\infty f(t)e^{-\alpha t}dt}{m\int_0^\infty te^{-\alpha t}dG(t)}.$$

LEMMA SM1.5 (Harris's lemma 4, p.163). *Suppose that* $m < 1$ *and* $\lim_{t\to\infty} f(t) = c$. *then* $K(t) \to \frac{c}{1-m}$.

Additional computation details for the proof of theorem 2.14. We detail how to obtain formula (49). We first take the Laplace transform of (39) for $\alpha = \lambda_c$ for $i = c+1$ and $j \in [\![c+1, J]\!]$. We distinguish the case $i = j$ from the others. If $j = c+1$, we obtain

$$\int_0^\infty M^a_{j,j}(t)e^{-\lambda_c t}dt = \frac{1}{\hat\rho^{(j)}(0)}\int_0^a \hat\rho^{(j)}(t)dt + 2p_S^{(j)}\int_0^\infty\left[\int_0^t d\mathcal{B}_j(t-u)M^a_{j,j}(u)du\right]e^{-\lambda_c t}dt.$$

By the Laplace transform property for the convolution, we deduce that

$$\int_0^\infty\left[\int_0^t d\mathcal{B}_j(t-u)M^a_{j,j}(u)du\right]e^{-\lambda_c t}dt = d\mathcal{B}_j^*(\lambda_c)\int_0^\infty M^a_{j,j}(t)e^{-\lambda_c t}dt,$$

hence

$$\int_0^\infty M^a_{j,j}(t)e^{-\lambda_c t}dt = \frac{1}{\hat\rho^{(j)}(0)}\int_0^a \hat\rho^{(j)}(t)dt + 2p_S^{(j)}d\mathcal{B}_j^*(\lambda_c)\int_0^\infty M^a_{j,j}(t)e^{-\lambda_c t}dt$$
$$= \frac{1}{\hat\rho^{(j)}(0)\times(1-2p_S^{(j)}d\mathcal{B}_j^*(\lambda_c))}\int_0^a \hat\rho^{(j)}(t)dt.$$

When $j > c+1$, we have:

$$\int_0^\infty M^a_{c+1,j}(t)e^{-\lambda_c t}dt = 2p_S^{(c+1)}d\mathcal{B}^*_{c+1}(\lambda_c)\int_0^\infty M^a_{c+1,j}(t)e^{-\lambda_c t}dt + 2p_L^{(c+1)}d\mathcal{B}_j^*(\lambda_c)\int_0^\infty M^a_{c+2,j}(t)e^{-\lambda_c t}dt.$$

Hence,

$$\int_0^\infty M^a_{c+1,j}(t)e^{-\lambda_c t}dt = \frac{2p_L^{(c+1)}}{1-2p_S^{(c+1)}d\mathcal{B}^*_{c+1}(\lambda_c)}\int_0^\infty M^a_{c+2,j}(t)e^{-\lambda_c t}dt.$$

Here, we obtain a recurrence formula between $\int_0^\infty M^a_{c+1,j}(t)e^{-\lambda_c t}dt$ and $\int_0^\infty M^a_{c+2,j}(t)e^{-\lambda_c t}dt$, and we obtain (49).

Second moments.

DEFINITION SM1.6. *Let* $a \geq 0$. *We define the second moment*

$$L^a(t) := \left(\mathbb{E}[(Y_t^{(a,j)})^2 | Z_0 = \delta_{i,0}]\right)_{i,j\in[\![1,J]\!]}.$$

LEMMA SM1.7. *$L^a(t)$ is solution of the renewal equation: $\forall (i,j) \in [\![1,J]\!]^2$,*

$$L^a_{i,j}(t) = \delta_{i,j}(1-\mathcal{B}_i(t))\mathbb{1}_{t\le a} + [2p_S^{(i)}L^a_{i,j} + 2p_L^{(i)}L^a_{i+1,j}] * d\mathcal{B}_i(t) + [2p^{(i)}_{2,0}(M^a_{i,j})^2 + 2p^{(i)}_{1,1}M^a_{i,j}M^a_{i+1,j} + 2p^{(i)}_{0,2}(M^a_{i+1,j})^2] * d\mathcal{B}_i(t). \tag{SM14}$$

Proof of lemma (SM1.7). Note that $\frac{\partial^2}{\partial s_j}F^{(i,a)}[s;t]|_{\mathbf{s}=1} = L^a_{i,j}(t) - M^a_{i,j}(t)$. We derive (SM12) with respect to s_j and obtain:

$$\frac{\partial^2}{\partial s_j^2}F^{(i,a)}[\mathbf{s};t] = \int_0^t \frac{\partial}{\partial s_j^2}f^{(i)}\big(F^a[\mathbf{s},u]\big)d\mathcal{B}_i(t-u)du$$

where

$$\begin{aligned}\frac{\partial^2}{\partial s_j^2}f^{(i)}\big(F^a[\mathbf{s},t]\big) &= 2p^{(i)}_{2,0}\left(F^{(i,a)}[\mathbf{s};t]\frac{\partial^2}{\partial s_j}F^{(i,a)}[\mathbf{s};t] + (\frac{\partial}{\partial s_j}F^{(i,a)}[\mathbf{s};t])^2\right)\\ &+ 2p^{(i)}_{0,2}\left(F^{(i+1,a)}[\mathbf{s};t]\frac{\partial^2}{\partial s_j}F^{(i+1,a)}[\mathbf{s};t] + (\frac{\partial}{\partial s_j}F^{(i+1,a)}[\mathbf{s};t])^2\right)\\ &+ p^{(i)}_{1,1}\Bigg(F^{(i+1,a)}[\mathbf{s};t]\frac{\partial^2}{\partial s_j}F^{(i,a)}[\mathbf{s};t] + 2\frac{\partial}{\partial s_j}F^{(i,a)}[\mathbf{s};t]\frac{\partial}{\partial s_j}F^{(i+1,a)}[\mathbf{s};t]\\ &\qquad + F^{(i,a)}[\mathbf{s};t]\frac{\partial^2}{\partial s_j}F^{(i+1,a)}[\mathbf{s};t]\Bigg).\end{aligned}$$

When $\mathbf{s}=1$, we get

$$\begin{aligned}L^a_{i,j}(t) - M^a_{i,j}(t) &= 2p^{(i)}_{2,0}\left(L^a_{i,j} - M^a_{i,j} + (M^a_{i,j})^2\right) * d\mathcal{B}_i(t)\\ &+ 2p^{(i)}_{0,2}\left(L^a_{i+1,j} - M^a_{i+1,j} + (M^a_{i+1,j})^2\right) * d\mathcal{B}_i(t)\\ &+ p^{(i)}_{1,1}\left(L^a_{i,j} - M^a_{i,j} + 2M^a_{i,j}M^a_{i+1,j} + L^a_{i+1,j} - M^a_{i+1,j}\right) * d\mathcal{B}_i(t).\end{aligned}$$

Using the system of equations (39), we deduce (SM14). ∎

THEOREM SM1.8. *Under the same hypotheses as in theorem 2.14, and supposing that for all $i \in [\![1,J]\!], \lambda_i > 0$, we have, for all $a \ge 0$:*

$$\forall i \in [\![1,J]\!], \quad \forall k \in [\![0,J-i]\!] \quad L^a_{i,i+k}(t) \sim \widetilde{L}_{i,i+k}(a)e^{2\lambda_{i,i+k}t}, \ \text{as } t\to\infty$$

such that

$$\widetilde{L}_{i,i}(a) = \frac{2p^{(i)}_{2,0}d\mathcal{B}^*_i(2\lambda_i)(\widetilde{M}^a_{i,i})^2}{1-2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_i)},$$

and for $k \in [\![1,J-i]\!]$,

$$\widetilde{L}_{i,i+k}(a) = \begin{cases} \frac{2p^{(i)}_{2,0}(\widetilde{M}^a_{i,i+k})^2 d\mathcal{B}^*_i(2\lambda_{i,i+k})}{1-2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_{i,i+k})} + l_{i,i+k}(a), & \text{if } \lambda_{i,i+k} \neq \lambda_i\\ \frac{2p^{(i)}_{2,0}(\widetilde{M}^a_{i,i+k})^2 d\mathcal{B}^*_i(2\lambda_{i,i+k})}{1-2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_{i,i+k})}, & \text{if } \lambda_{i,i+k} = \lambda_i\end{cases} \tag{SM15}$$

where

$$l_{i,i+k}(a) = \frac{\left[\widetilde{L}_{i+1,i+k}(a) + 2p^{(i)}_{1,1}\widetilde{M}^a_{i,i+k}\widetilde{M}^a_{i+1,i+k} + 2p^{(i)}_{0,2}(\widetilde{M}^a_{i+1,i+k})^2\right]d\mathcal{B}^*_i(2\lambda_{i,i+k})}{1-2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_{i,i+k})}.$$

Proof. Let $a \geq 0$. We introduce the following notations

$$\widehat{L}^a_{i,i+k}(t) = L^a_{i,i+k}(t) e^{-2\lambda_{i,i+k} t}, \quad \widehat{d\mathcal{B}_i}(t) = \frac{d\mathcal{B}_i(t)}{d\mathcal{B}^*_i(2\lambda_{i,i+k})} e^{-2\lambda_{i,i+k} t}.$$

We use the same approach as that performed for the proof of theorem 2.14, and proceed by recurrence:

$$\mathcal{H}^k : \quad \forall i \in [\![1, J-k]\!], \; L^a_{i,i+k}(t) \sim \widetilde{L}^a_{i,i+k} e^{2\lambda_{i,i+k} t}, \text{ as } t \to \infty.$$

When $k = 0$, according to (SM14) $L^a_{i,i}$ is solution of the renewal equation:

$$L^a_{i,i}(t) = (1 - \mathcal{B}_i(t)) \mathbb{1}_{t \leq a} + 2p^{(i)}_{2,0} (M^a_{i,i})^2 * d\mathcal{B}_i(t) + 2p^{(i)}_S L^a_{i,i} * d\mathcal{B}_i(t). \tag{SM16}$$

We rescale (SM16) by $e^{-2\lambda_i t}$ and obtain:

$$\widehat{L}^a_{i,i}(t) = e^{-2\lambda_i t} \left[(1 - \mathcal{B}_i(t)) \mathbb{1}_{t \leq a} + 2p^{(i)}_{2,0} (M^a_{i,i})^2 * d\mathcal{B}_i(t) \right] + 2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_i) \widehat{L}^a_{i,i} * \widehat{d\mathcal{B}_i}(t).$$

Note that as $2\lambda_i > \lambda_i > 0$, we have $2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_i) < 1$, so that we can use lemma SM1.5. We compute the limit of the source term :

$$\lim_{t \to \infty} e^{-2\lambda_i t} \left[(1 - \mathcal{B}_i(t)) \mathbb{1}_{t \leq a} + 2p^{(i)}_{2,0} (M^a_{i,i})^2 * d\mathcal{B}_i(t) \right].$$

From hypothesis 2.2, we have:

$$\int_0^\infty (1 - \mathcal{B}_i(t)) \mathbb{1}_{t \leq a} e^{-\lambda_i t} dt \leq \frac{1}{\bar{b}_i} \int_0^\infty d\mathcal{B}_i(t) e^{-\lambda_i t} dt < \infty.$$

Thus, $(1 - \mathcal{B}_i(t)) \mathbb{1}_{t \leq a} e^{-\lambda_i t} \in \mathbf{L}^1(\mathbb{R}_+)$ and, $\lim_{t \to \infty} e^{-\lambda_i t} [1 - \mathcal{B}_i(t)] = 0$. Using the hypothesis $\lambda_i > 0$, we obtain that $\lim_{t \to \infty} e^{-2\lambda_i t} [1 - \mathcal{B}_i(t)] = 0$. Then,

$$e^{-2\lambda_i t} (M^a_{i,i})^2 * d\mathcal{B}_i(t) = \int_0^\infty \mathbb{1}_{[0,t]} (M^a_{i,i}(t-u) e^{-\lambda_i (t-u)})^2 d\mathcal{B}_i(u) e^{-2\lambda_i u} du.$$

Using theorem 2.14, we have $M^a_{i,i}(t) \sim e^{\lambda_i t} \widetilde{M}_{i,i}(a)$, as $t \to \infty$. Applying Lebesgue dominated convergence theorem, we obtain

$$\lim_{t \to \infty} e^{-2\lambda_i t} (M^a_{i,i})^2 * d\mathcal{B}_i(t) = (\widetilde{M}_{i,i}(a))^2 d\mathcal{B}^*_i(2\lambda_i).$$

Then, applying lemma SM1.5, we deduce:

$$L^a_{i,i}(t) \sim \widetilde{L}_{i,i}(a) e^{2\lambda_i t}, \text{ as } t \to \infty, \text{ where } \widetilde{L}_{i,i}(a) = \frac{2p^{(i)}_{2,0} d\mathcal{B}^*_i(2\lambda_i) (\widetilde{M}_{i,i}(a))^2}{1 - 2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_i)}.$$

Hence, $\mathcal{H}^0$ is true. Then, we suppose that $\mathcal{H}^{k-1}$ holds and we show $\mathcal{H}^k$. According to (SM14), we write the equation for $L^a_{i,i+k}$ and rescale it by $e^{-2\lambda_{i,i+k} t}$:

$$\begin{aligned}
\widehat{L}^a_{i,i+k}(t) &= 2p^{(i)}_S d\mathcal{B}^*_i(2\lambda_{i,i+k}) \widehat{L}^a_{i,i+k} * \widehat{d\mathcal{B}_i}(t) + e^{-2\lambda_{i,i+k} t} 2p^{(i)}_L L^a_{i+1,i+k} * d\mathcal{B}_i(t) \\
&+ e^{-2\lambda_{i,i+k} t} \left[2p^{(i)}_{2,0} (M^a_{i,i+k})^2 + 2p^{(i)}_{1,1} M^a_{i,i+k} M^a_{i+1,i+k} + 2p^{(i)}_{0,2} (M^a_{i+1,i+k})^2 \right] * d\mathcal{B}_i(t).
\end{aligned}$$

Here, $m = 2p_S^{(i)} d\mathcal{B}_i^*(2\lambda_{i,i+k}) < 1$, so that we can use lemma SM1.5. We first compute the limit of $e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t)$ when t goes to infinity when either $\lambda_{i,i+k} = \lambda_i$ or $\lambda_{i,i+k} \neq \lambda_i$. We start with the case $\lambda_{i,i+k} \neq \lambda_i$ (so, $\lambda_{i,i+k} = \lambda_{i+1,i+k}$). For all $t \geq 0$, we have:

$$e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t) = d\mathcal{B}_i^*(2\lambda_{i,i+k}) \int_0^\infty \mathbb{1}_{[0,t]} e^{-2\lambda_{i,i+k}(t-u)} L^a_{i+1,i+k}(t-u) \widehat{d\mathcal{B}_i}(u) du.$$

According to $\mathcal{H}^{k-1}$, we know that $L^a_{i+1,i+k}(t) \sim \widetilde{L}_{i+1,i+k}(a) e^{2\lambda_{i,i+k}t}$. We deduce with a Lebesgue dominated convergence theorem that:

$$\lim_{t\to\infty} e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t) = d\mathcal{B}_i^*(2\lambda_{i,i+k}) \widetilde{L}_{i+1,i+k}(a).$$

We apply the same method as above for the other terms of the source term. Theorem 2.14 gives us that $M^a_{i,i+k} \sim e^{\lambda_{i,i+k}t} \widetilde{M}_{i,i+k}(a)$ and $M^a_{i+1,i+k} \sim e^{\lambda_{i,i+k}t} \widetilde{M}_{i+1,i+k}(a)$. Using Lebesgue dominated convergence theorem, we obtain:

$$\begin{aligned} \lim_{t\to\infty} e^{-2\lambda_{i,i+k}t} &\left[2p^{(i)}_{2,0} (M^a_{i,i+k})^2 + 2p^{(i)}_{1,1} M^a_{i,i+k} M^a_{i+1,i+k} + 2p^{(i)}_{0,2} (M^a_{i+1,i+k})^2 \right] * d\mathcal{B}_i(t) \\ &= \Big[2p^{(i)}_{2,0} (\widetilde{M}_{i,i+k}(a))^2 + 2p^{(i)}_{1,1} \widetilde{M}_{i,i+k}(a) \widetilde{M}_{i+1,i+k}(a) \\ &\qquad + 2p^{(i)}_{0,2} (\widetilde{M}_{i+1,i+k}(a))^2 \Big] d\mathcal{B}_i^*(2\lambda_{i,i+k}). \end{aligned}$$

We then consider the case $\lambda_{i,i+k} = \lambda_i > \lambda_{i+1,i+k}$ and start by computing the limit of $e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t)$.

$$e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t) = d\mathcal{B}_i^*(2\lambda_{i,i+k}) e^{-2(\lambda_{i,i+k}-\lambda_{i+1,i+k})t} \int_0^\infty \mathbb{1}_{[0,t]} e^{-2\lambda_{i+1,i+k}(t-u)} L^a_{i+1,i+k}(t-u) \widehat{d\mathcal{B}_i}(u) du.$$

Using $\mathcal{H}^{k-1}$ and Lebesgue dominated convergence theorem, we first obtain that

$$\lim_{t\to\infty} \int_0^\infty \mathbb{1}_{[0,t]} e^{-2\lambda_{i+1,i+k}(t-u)} L^a_{i+1,i+k}(t-u) \widehat{d\mathcal{B}_i}(u) du = d\mathcal{B}_i^*(2\lambda_{i+1,i+k}) \widetilde{L}_{i+1,i+k}(a) < \infty,$$

hence,

$$\lim_{t\to\infty} e^{-2\lambda_{i,i+k}t} L^a_{i+1,i+k} * d\mathcal{B}_i(t) = 0.$$

Then, theorem 2.14 give us that $M^a_{i,i+k} \sim e^{\lambda_{i,i+k}t} \widetilde{M}_{i,i+k}(a)$ and $M^a_{i+1,i+k} \sim e^{\lambda_{i+1,i+k}t} \widetilde{M}_{i+1,i+k}(a)$. Using similar methods, we obtain:

$$\begin{aligned} \lim_{t\to\infty} e^{-2\lambda_{i,i+k}t} &\left[2p^{(i)}_{2,0} (M_{i,i+k})^2 + 2p^{(i)}_{1,1} M_{i,i+k} M_{i+1,i+k} + 2p^{(i)}_{0,2} (M_{i+1,i+k})^2 \right] * d\mathcal{B}_i(t) \\ &= 2p^{(i)}_{2,0} (\widetilde{M}_{i,i+k})^2 d\mathcal{B}_i^*(2\lambda_{i,i+k}). \end{aligned}$$

We conclude by applying lemma SM1.5 that $\mathcal{H}^k$ holds. ∎

Variance.

DEFINITION SM1.9. *We write $v_j^a(t)$, the variance of $Y_t^{(j,a)}$ starting from a mother cell on the first layer such that:*

$$v_j^a(t) = \mathbb{E}[(Y_t^{(j,a)})^2 | Z_0 = \delta_{1,0}] - \mathbb{E}[Y_t^{(j,a)} | Z_0 = \delta_{1,0}]^2. \tag{SM17}$$

We study the asymptotic behavior of the variance $v_j^a(t)$ when the first layer is the leading one.

COROLLARY SM1.10. *Let $a \geq 0$. Under the same hypotheses as in theorem SM1.8 and supposing that $c = 1$, we have*

$$\forall k \in [\![1, J]\!] \quad v_j^a(t) \sim \widetilde{v}_j(a) e^{2\lambda_c t}, \text{ as } t \to \infty$$

where

$$\widetilde{v}_j(a) = \widetilde{L}_{1,j}(a) - (m_j(a))^2 = \left[\frac{2p_{2,0}^{(1)} d\mathcal{B}_1^*(2\lambda_1)}{1 - 2p_S^{(1)} d\mathcal{B}_1^*(2\lambda_1)} - 1 \right] (m_j(a))^2.$$

Proof. Let $a \geq 0$. According to theorem SM1.8 and using that $c = 1$, we have:

$$\forall j \in [\![1, J]\!], \quad L_{1,j}^a(t) \sim \widetilde{L}_{1,j}(a) e^{2\lambda_1 t}, \quad \text{as } t \to \infty\,.$$

Using theorem 2.14 and SM1.8, we deduce for all $j \in [\![1, J]\!]$:

$$\widetilde{v}_j(a) = \widetilde{L}_{1,j}(a) - (m_j(a))^2 = \left[\frac{2p_{2,0}^{(1)} d\mathcal{B}_1^*(2\lambda_1)}{1 - 2p_S^{(1)} d\mathcal{B}_1^*(2\lambda_1)} - 1 \right] (m_j(a))^2.$$

∎

SM2. Numerical simulation procedures.

SM2.0.1. Stochastic simulation procedures.

Markov case. Considering a markovian case, we simulate the process Z_t solution of the SDE (2) with the Gillespie algorithm. We use the package StochSS [SM2]. We consider that for each layer $j \in [\![1, 3]\!]$, $p_{1,1}^{(j)} = 0$. Hence, $p_{2,0}^{(j)} = p_S^{(j)}$ and $p_{0,2}^{(j)} = 1 - p_S^{(j)}$. Considering a system with 4 layers, our system is ruled by the 7-th reactions below:

$$\begin{aligned}
\mathcal{R}_1: &\quad N_1 \to N_1 + N_1 &&\text{with rate } b_1 p_S^{(1)},\\
\mathcal{R}_2: &\quad N_1 \to N_2 + N_2 &&\text{with rate } b_1(1 - p_S^{(1)}),\\
\mathcal{R}_3: &\quad N_2 \to N_2 + N_2 &&\text{with rate } b_2 p_S^{(2)},\\
\mathcal{R}_4: &\quad N_2 \to N_3 + N_3 &&\text{with rate } b_2(1 - p_S^{(2)}),\\
\mathcal{R}_5: &\quad N_3 \to N_3 + N_3 &&\text{with rate } b_3 p_S^{(3)},\\
\mathcal{R}_6: &\quad N_3 \to N_4 + N_4 &&\text{with rate } b_3(1 - p_S^{(3)}),\\
\mathcal{R}_7: &\quad N_4 \to N_4 + N_4 &&\text{with rate } b_4.
\end{aligned}$$

General case. We simulate our process using the algorithm SM1, on a predefine time horizon $T_{\max}$.

Algorithm SM1 Simulation stochastic process

1: Define a sequence $\mathcal{S}$ of cells of a given age and layer.
2: Simulate the time of division of each cell in $\mathcal{S}$
3: **while** $t < T_{\max}$ **do**
4: Select the next cell m that will divide. l^m is its layer index and l^m is the age at division.
5: Randomly draw the layer of its daughters cell l^{d_1} and l^{d_2} according to the probabilities $p_{2,0}^{(l^m)}$,$p_{1,1}^{(l^m)}$ and $p_{2,0}^{(l^m)}$.
6: Randomly draw the next time of division of daughter cell d_1 according to its layer index l^{d_1}.
7: Randomly draw the next time of division of daughter cell d_2 according to its layer index l^{d_2}.
8: Add d_1 and d_2 into the sequence $\mathcal{S}$
9: $t \leftarrow t + t^m$
10: **end while**

SM2.0.2. Deterministic simulation protocol. To solve numerically the problem (3), we design a dedicated finite volume scheme adapted to the non-conservative form with proper boundary conditions. We define the time step Δt and the age step Δa. The time discretization is defined by

$$t_0 = 0, \quad t_{n+1} = t_n + \Delta t, \quad \text{for } n = 0, ..., N_t$$

where $(N_t + 1)\Delta t$ is the time horizon of the simulation. Similarly, N_a is the number of cells[1] in the domain. The cells $\mathcal{C}_i$ are indexed by a rational number i ($\frac{1}{2}$,$\frac{3}{2}$, etc.) with $i \in [\![\frac{1}{2}, N_a - \frac{1}{2}]\!]$. The edges of each cell are located at $a_{i-\frac{1}{2}} = (i - \frac{1}{2})\Delta a$ and $a_{i+\frac{1}{2}} = (i + \frac{1}{2})\Delta a$ (remark that $\Delta a = a_{i+\frac{1}{2}} - a_{i-\frac{1}{2}}$ and $a_0 = 0$). As age and time evolve at the same speed, we chose N_a such that $t_{N_t} - a^0_{\max} < N_a \Delta a$ where $a^0_{\max}$ is the maximal age of the initial distribution.

Let $j \in [\![1, J]\!]$. We define $P^j_{n,i}$ as the mean value of the density $\rho^{(j)}$ in cell $\mathcal{C}_i$ at time t_n:

$$P^j_{n,i} := \frac{1}{\Delta a} \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t_n, a)\, da\,.$$

We integrate the equation $\partial_t \rho^{(j)} + \partial_a \rho^{(j)} = -b_j \rho^{(j)}$ with respect to age in cell $\mathcal{C}_i$ and obtain:

$$\frac{d}{dt} \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t, a)\, da = -\rho^{(j)}(t, a_{i+\frac{1}{2}}) + \rho^{(j)}(t, a_{i-\frac{1}{2}}) - \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} b_j(a) \rho^{(j)}(t, a)\, da.$$

Then, we suppose that all b_js functions are regular enough so that we can approximate b_j, for all $j \in [\![1, J]\!]$ on each cell $\mathcal{C}_i$ by their mean value $\bar{b}^i_j$. We obtain:

$$\frac{d}{dt} \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t, a)\, da = -\rho^{(j)}(t, a_{i+\frac{1}{2}}) + \rho^{(j)}(t, a_{i-\frac{1}{2}}) - \bar{b}^i_j \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t, a)\, da.$$

[1]The cell is here the standard name used for each elementary volume in the framework of finite volume methods.

We approximate the derivative in time with a finite difference scheme:

$$\frac{d}{dt}\int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t_n,a)\,da = \frac{1}{\Delta t}\left[\int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t_{n+1},a)da - \int_{a_{i-\frac{1}{2}}}^{a_{i+\frac{1}{2}}} \rho^{(j)}(t_n,a)da\right] + \mathcal{O}(\Delta t)$$

and we deduce:

$$\frac{\Delta a}{\Delta t}\left[P^j_{n+1,i} - P^j_{n,i}\right] = -\rho^{(j)}(t,a_{i+\frac{1}{2}}) + \rho^{(j)}(t,a_{i-\frac{1}{2}}) - \bar{b}^i_j \Delta a P^j_{n,i}.$$

The edge terms $\rho^{(j)}(t,a_{i+\frac{1}{2}})$ and $\rho^{(j)}(t,a_{i-\frac{1}{2}})$ correspond to the fluxes that cross the boundaries of cell $\mathcal{C}_i$. When $i=\frac{1}{2}$, the boundary condition of equation (3) gives us the value of this term:

$$\begin{aligned}\rho^{(j)}(t_n,a_0) &= 2p_S^{(j)}\int_0^\infty b_j(a)\rho^{(j)}(t_n,a)da + 2(1-p_S^{(j-1)})\int_0^\infty b_{j-1}(a)\rho^{(j-1)}(t_n,a)da\\ &= 2p_S^{(j)}\sum_i\int_{\mathcal{C}_i} b_j(a)\rho^{(j)}(t_n,a)da + 2(1-p_S^{(j-1)})\sum_i\int_{\mathcal{C}_i} b_{j-1}(a)\rho^{(j-1)}(t_n,a)da\\ &= 2p_S^{(j)}\Delta t\sum_i \bar{c}_i P^j_{n,i} + 2(1-p_S^{(j-1)})\Delta t\sum_i \bar{c}^{j-1}_i P^{j-1}_{n,i}\,.\end{aligned}$$

When $i\neq\frac{1}{2}$, we approximate each term $\rho^{(j)}(t_n,a_{i+\frac{1}{2}})$ by

$$\rho^{(j)}(t_n,a_{i+\frac{1}{2}}) = P^j_{n,i+\frac{1}{2}} + \mathcal{O}(\Delta a).$$

Hence, we obtain the following numerical scheme:

$$P^j_{n+1,i} = \left[1 - \bar{b}^j_i\Delta t - \frac{\Delta t}{\Delta x}\right]P^j_{n,i} + \frac{\Delta t}{\Delta x}P^j_{n,i-1}$$

$$P^j_{n+1,\frac{1}{2}} = \left[1 - \bar{c}^j_{\frac{1}{2}}\Delta t - \frac{\Delta t}{\Delta x}\right]P^j_{n,\frac{1}{2}} + 2s_j\Delta t\sum_i \bar{c}_i P^j_{n,i} + 2(1-s_{j-1})\Delta t\sum_i \bar{c}^{j-1}_i P^{j-1}_{n,i}.$$

SM2.1. Construction of figure 4. In this part, we give some details about the construction of figure 4. We simulate the SDE (2) using the algorithm SM1 and the PDE (3) using the algorithm described in the subsection below (see SM2.0.2) taking $\Delta a = 9.5\times 10e-3$ and $\Delta t = 10e-4$.

We discretized the age according to a sequence of integers $k\in[\![1,50]\!]$. Let $j\in[\![1,J]\!]$ be a layer index. The color bar associated with age k for the j-th layer corresponds to the total number of cells on the j-th layer of age $a\in[k,k+1[$ renormalized by the total number of cells:

$$\frac{<< Z_t, \mathbb{1}_{j,k\le a<k+1} >>}{<< Z_t,\mathbb{1} >>}\,.$$

The dashed black line with the age k for the j-th layer corresponds to:

$$\frac{\int_k^{k+1} \rho^{(j)}(t,a)da}{\sum_{j=1}^4 \int_0^{+\infty} \rho^{(j)}(t,a)da} \sim \frac{\displaystyle\sum_{i=\lfloor \frac{k}{\Delta x} \rfloor}^{\lfloor \frac{k+1}{\Delta x} \rfloor -1} P_{n,i}^j}{\displaystyle\sum_{j=1}^4 \sum_i P_{n,i}^j}.$$

The color solid lines which represent the stable distribution $\hat{\rho}$ and compute their value at each age point k by

$$\frac{\int_k^{k+1} \hat{\rho}^{(j)}(a)da}{\sum_{j=1}^4 \int_0^{+\infty} \hat{\rho}^{(j)}(a)da}.$$

SM2.2. Parameter estimation procedure. Using the software D2D [SM7], we estimate the parameters of our model, using an additive Gaussian noise statistical model (standard least squares likelihood). The standard deviation and the initial number N of cells on the first layer are also estimated. To investigate the practical identifiability, we compute the profile likelihood estimate (PLE) [SM6]. We observe that all the parameters are practically identifiable except the probability of staying on the second layer $p_S^{(2)}$ (see Figure SM1a). In contrast, most of the parameters are not practically identifiable when we consider the total number of cells as the observable function ($\sigma(t;p) = \sum_{j=1}^J M^{(j)}(t;p)$, Figure SM1b).

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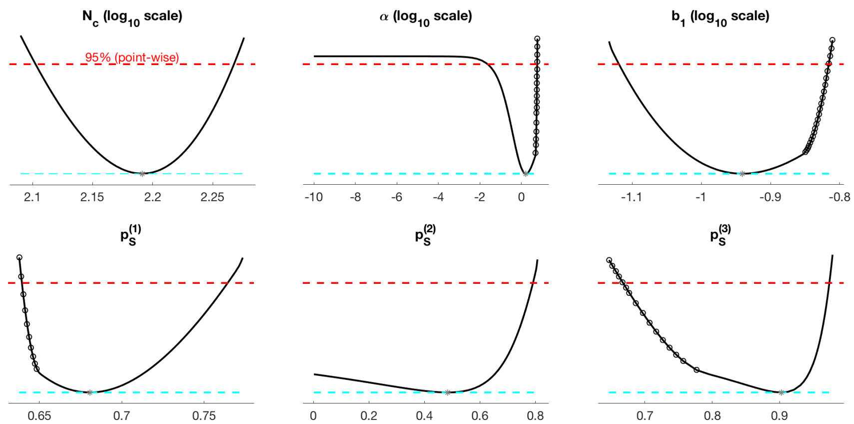
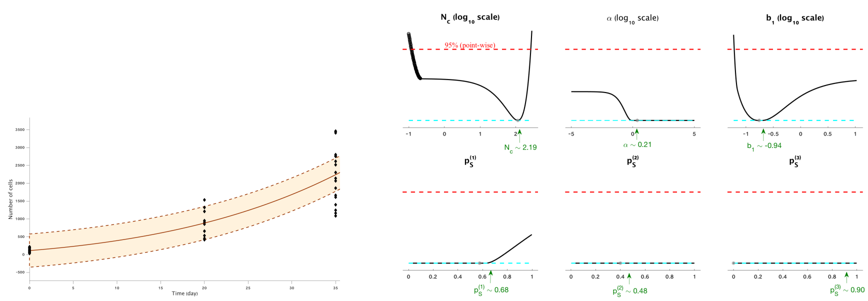


(a) Practical Identifiability for $\sigma(t;p) = (M^{(j)}(t;p))_{j\in[\![1,J]\!]}$



(b) Practical Identifiability for $\sigma(t;p) = \sum_{j=1}^{J} M^{(j)}(t;p)$

FIGURE SM1. ***Practical Identifiability***. ***Figure SM1a*** *Profile likelihood estimate (PLE) for each parameter in the set* $\mathbf{P}_{exp} = \{N, b_1, \alpha, p_S^{(1)}, p_S^{(2)}, p_S^{(3)}\}$ *when the observation function is* $\sigma(t;p) = (M^{(j)}(t;p))_{j\in[\![1,J]\!]}$*.The red dashed lines correspond to the* 95%*-statistical threshold while the blue dashed lines correspond to the optimum value of the likelihood.* ***Figure SM1b*** *Parameter estimation results for* $\sigma(t;p) = \sum_{j=1}^{J} M^{(j)}(t;p)$*.* ***Left panel:*** *Data fitting model with model (8). The black diamonds represent the experimental data (total number of cells), the solid line is the best fit solution of (8) and the dashed lines are drawn from the estimated variance.* ***Left panel:*** *Profile likelihood estimates of each parameter in the set* P_{exp}*.*