INITIAL-BOUNDARY VALUE PROBLEM TO THE LIFSHITZ-SLYOZOV EQUATION WITH NON-SMOOTH RATES AT THE BOUNDARY

JUAN CALVO*, ERWAN HINGANT†, AND ROMAIN YVINEC‡,§

Abstract. We prove existence and uniqueness of solutions to the initial-boundary value problem for the Lifshitz–Slyozov equation (a non-linear transport equation on the half-line) focusing on the case of kinetic rates that are not Lipschitz continuous at the origin. Our theory covers in particular those cases with rates behaving as power laws at the origin, for which an inflow behaviour is expected and a boundary condition describing nucleation phenomena needs to be imposed. The method we introduce here to prove existence is based on a mild formulation with a careful analysis on the behavior of characteristics near the singular boundary. Uniqueness exploits monotonicity properties of the associated transport equation.

Contents

1. Introduction 2
1.1. The Lifshitz-Slyozov equation 2
1.2. Definitions and main results 4
1.3. Outline 8
2. Characteristics curves and linear problem 8
2.1. Characteristics: definition and construction of the associated flow 9
2.2. Diffeomorphism through the characteristic curves 12
2.3. Mild formulation and regularities 14
3. The non-linear problem 19
3.1. Existence of solutions 19
3.2. Uniqueness 22
3.3. Criteria for global and local solutions 27
4. Annex 29
4.1. Proof of Lemma 2.7 29
4.2. Proof of Proposition 2.8 29
4.3. Proof of Proposition 2.9 32

---

*Departamento de Matemática Aplicada and Research Unit "Modeling Nature" (MNat), Universidad de Granada, Granada, Spain.

†Departamento de Matemática, Universidad del Bío-Bío, Concepción, Chile.

‡PRC, INRA, CNRS, IFCE, Université de Tours, 37380 Nouzilly, France.

§Associate researcher at Cogitamus laboratory.

E-mail addresses: juancalvo@ugr.es, ehingant@ubiobio.cl, romain.yvinec@inrae.fr.

Date: April 3, 2020.

2010 *Mathematics Subject Classification.* 35A01, 35B60, 35C99, 35L04, 35M13, 35Q92.

Key words and phrases. non-linear transport equation, singular initial-boundary value problem, dynamic boundary condition, characteristics and mild formulation, Ostwald ripening, nucleation theory, polymerization.

Acknowledgments 37
References 37

1. Introduction

1.1. **The Lifshitz-Slyozov equation.** The purpose of this work is to provide a well-posedness theory for the Lifshitz–Slyozov model with inflow boundary conditions under widely general assumptions on the initial data and the kinetic rates. The Lifshitz–Slyozov system [26] describes the temporal evolution of a mixture of monomers and aggregates, where individual monomers can attach to or detach from already existing aggregates. The aggregate distribution follows a transport equation with respect to a size variable, whose transport rates are coupled to the dynamics of monomers through a mass conservation relation. The initial-boundary value problem for the Lifshitz–Slyozov model thus reads

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} + \dfrac{\partial[(a(x)u(t)-b(x))f(t,x)]}{\partial x} = 0\,, \quad t>0\,,\ x\in(0,\infty)\,, \\ u(t) + \displaystyle\int_0^\infty x f(t,x)\,dx = \rho\,, \quad t>0 \end{cases} \tag{1.1}$$

for some given $\rho > 0$, subject to the initial condition

$$f(0,x) = f^{in}(x)\,, \qquad x\in(0,\infty) \tag{1.2}$$

and the boundary condition

$$\lim_{x\to 0^+} (a(x)u(t)-b(x))f(t,x) = \mathfrak{n}(u(t))\,, \quad t>0 \tag{1.3}$$

whenever $u(t) > \lim_{x\to 0^+} \frac{b(x)}{a(x)}$. Here $f(t,x)$ is a non-negative distribution of aggregates according to their size x and time t, $u(t)$ is the monomer concentration and ρ is interpreted as the total mass of the system. The kinetic rates $a(x)$ and $b(x)$ determine how fast do attachment (a given monomer attaches to a given aggregate) and detachment (a monomer detaches from a given aggregate) reactions take place. Aggregates change their size over time according to the quantity of monomers that they gain or lose through the previous reactions. Note that the attachment process is a second order kinetics whereas detachment is a first order kinetics, as reflected in the transport term in (1.1).

The Lifshitz–Slyozov model has been traditionally used to describe late stages of phase transitions, where Ostwald ripening phenomena take place: large aggregates grow larger at the expense of smaller ones, in which case a boundary condition like (1.3) is not needed; recall indeed that the classical Lifshitz–Slyozov rates are given by $a(x) = x^{1/3}$ and $b(x) = 1$, see e.g. [29]. In standard nucleation theory, a discrete size model analog, named the Becker-Döring model [19], is rather used to describe the initial stage of phase transition, where the nucleation process is the dominant one. Recently, the intermediate stage has been considered in the physics literature [1, 2, 27, 33, 34], where the growth of large aggregates and the ongoing nucleation rate are of equal importance, leading to equations like (1.1)–(1.3) or variants of it. Indeed, some sets of kinetic rates for Eq. (1.1) may lead to Ostwald ripening phenomena only after a certain transient period, where the dynamics of the Lifshitz–Slyozov model are driven by boundary effects at very small sizes, and for which the boundary term (1.3) becomes important. Moreover, recent applications of this framework in biologically oriented contexts utilize a different set of kinetic rates and then a boundary condition becomes mandatory in order to make sense of the model. A growing literature can be found on applications to protein polymerization phenomena and neurodegenerative diseases, starting from the so-called prion model and some of its variants (see e.g. [4, 14, 17, 24, 25, 31, 35] and references therein), whose different versions come as modifications of the standard Lifshitz–Slyozov equations. Inflow boundary conditions are used to describe nucleation processes; the discrete models considered in [11, 7] are also related to this scenario by means of suitable scaling limits as we mention below. We also have in mind applications to modeling in Oceanography. For instance, the sea-surface microlayer (see e.g. [39]) is rich in conglomerates that grow in size by an aggregation process whereby particulate organic carbon attaches to transparent exopolymeric particles; detachment effects can also take place and eventually additional terms may be included in (1.1)–(1.3), e.g. coagulation integrals. Tentative applications of variants of (1.1)–(1.3) can be also envisioned where x is a depth variable and gradual sinking of aggregates ("marine snow" [3, 20]) proceeds by a ballasting process. We conjecture that more applications of this framework will gradually arise. The common feature is that the boundary condition (1.3) can be interpreted as the synthesis of new aggregates from monomers and not necessarily by means of a mass action law kinetics.

To the best of our knowledge, works covering mathematical aspects of the initial-boundary value problem for the Lifshitz–Slyozov model are presently scarce. We mention here [11], where the model (1.1)–(1.3) is deduced as a scaling limit of the Becker–Döring model and the inflow boundary condition is interpreted in terms of the scaling and the mesoscopic reaction rates; note that some partial analysis in this direction were already given in [7]. We also mention [4], where it is shown that in some particular cases the model leads to dust formation (concentration at zero size), a behavior that can be somewhat prevented if fragmentation terms are incorporated into the model. Incidentally, the model with kinetic rates such that the boundary becomes characteristic is considered in [6]. Quite the contrary, the mathematical literature for the classical Lifshitz–Slyozov model is well established. Concerning density solutions, existence and uniqueness of mild solutions for Lipschitz rate functions is given in [6], whereas existence and uniqueness of weak solutions for rates not necessarily regular at the origin are provided in [21]. Measure solutions were considered in [6, 29, 30]. Mathematical justifications of the connection between the Becker–Döring model and the Lifshitz–Slyozov model can be found in [7, 23, 28, 32]; the results therein can be also understood as existence proofs. The long time behavior is analyzed in [6, 8], however our understanding of the dynamical behavior is not complete yet. Therefore, numerical simulations are a useful way to get further insights on the asymptotic behavior; some contributions along these lines are [5, 15]. A number of variants of the Lifshitz–Slyozov model have been considered in the literature as well; we refer to [38, 9, 18, 37, 16] for diffusive versions (also advocate to represent intermediate stages of aggregates growth) and to [22, 29, 30] for the Lifshitz–Slyozov–Wagner model.

In this contribution we study existence and uniquenes of local-in-time solutions for (1.1)–(1.3), together with continuation criteria and results on long-time behavior. In order to tackle the well-posedness of (1.1)–(1.3) we have chosen to use techniques based on characteristics and mild solutions. This has the advantage of providing a semi-explicit representation formula (which may prove useful for e.g. designing particle methods) and is reminiscent of the works [6, 8]. Due to the wide spectrum of applications mentioned above, it is crucial to be able to cope with rates that are not regular at the origin. This generates a number of technical difficulties in order to make sense of characteristic curves, difficulties that are not present when the rates are globally Lipschitz; one of the main contributions of this paper is to provide a reformulation that allows to give a suitable meaning to phase space trajectories/characteristic curves even when there is no forward-in-time uniqueness for those. We take definite advantage of working in "spatial" dimension one and represent solutions as a mixture of trajectories reaching the initial configuration or the boundary datum respectively, for every time instant. In such a way we are able to construct mild solutions unambiguously. Similar ideas belong to the folklore on boundary problems for transport equations, although we have not been able to find a suitable reference covering our non-Lipschitz regularity setting. Note in particular that we do not assume to have transport fields with bounded divergence; recall that the assumptions in [13] can be lifted in some cases, see e.g. [10, 12]. Thus, for reader's convenience we work out the full theory from scratch, which we believe to be of independent interest for the sake of other applications. Our construction guarantees that no singularities (shock formation, concentration phenomena) are created during the temporal evolution despite of the incoming boundary flow. We also extend the uniqueness proof in [21] to be able to cope with inflow solutions in this low-regularity context. As regards the scope of the theory we develop here, we provide examples of local solutions that can be extended to global ones and at the same time we clearly show why local-in-time existence of inflow solution is the best we can hope for generically. The breakdown of global existence is proved by giving examples of solutions that do not exist globally in time because the boundary condition loses its meaning, which raises the problem of giving a wider meaning to the solution concept in order to be able to extend every local solution to a global one. This is an important issue that is deeply connected with a full understanding of the long time behavior and will be tackled elsewhere by the authors and collaborators.

1.2. **Definitions and main results.** Let us recall a few classical notations. Given a subset Ω of $\mathbb{R}^d$ equipped with the subspace topology, we denote by $\mathcal{C}^k(\Omega)$ the space of continuous real-valued function defined on Ω with at least k continuous derivatives and $\mathcal{C}^k_c(\Omega)$ its subspace consisting of compactly supported functions. For a measure μ defined on the borelian sets of Ω we understand by $L^1(\Omega,\mu)$, resp. $L^\infty(\Omega,\mu)$, the classical Lebesgue space consisting of the equivalent class of μ-integrable, resp. μ-essentially bounded, real-valued functions defined on Ω agreeing μ-almost everywhere (*a.e.*). The reference to the measure μ might be omitted if we clearly refer to the Lebesgue measure. We will make used of two more spaces, for X a Banach space and I an interval: $\mathcal{C}(I, w-X)$ denoted the space of continuous X-valued functions defined on I, where X is endowed with its weak topology and $L^\infty(I,X)$ the Bochner space of essentially bounded X-valued functions defined on I agreeing *a.e.* with respect to the Lebesgue measure on I. Also, during the document we will use a notation like $C(A,B,\dots)$ to denote a positive constant depending on the quantities between brackets, whose actual value is not relevant. Its value may change from line to line without explicit mention. Finally, for any

$T \in (0,\infty]$ we set

$$\Omega_T = [0,T) \times (0,\infty) \text{ and } \Omega_T^* = (0,T) \times (0,\infty).$$

Note that when referring to $T > 0$ in what follows we shall always assume it is finite unless it is explicitly stated otherwise.

Definition 1.1 (Kinetic rates)**.** A triplet $\{a, b, \mathfrak{n}\}$ defines kinetic rates provided that:

i) a and b are locally bounded and non-negative functions on $[0,\infty)$,
ii) The ratio function $\Phi(x) \coloneqq b(x)/a(x)$ is defined for *a.e.* $x > 0$ in $[0,\infty]$ and has a limit $\Phi_0 \in [0,\infty]$ at 0^+, which we call the threshold,
iii) $\mathfrak{n}$ is a locally bounded and non-negative function on $[\Phi_0,\infty)$.

Definition 1.2 (Solution to the initial - boundary value problem)**.** Let $T \in (0,\infty]$. Assume to be given the kinetic rates $\{a, b, \mathfrak{n}\}$, a constant $\rho > 0$ and a non-negative function $f^{\rm in}$ belonging to $L^1(0,\infty)$. We say that a non-negative function f defined on Ω_T is a solution to the Lifshitz–Slyozov equation (1.1) on $[0,T)$ with mass ρ, kinetic rates $\{a, b, \mathfrak{n}\}$ and initial value $f^{\rm in}$ if the following statements are satisfied:
i) The function f belongs to $\mathcal{C}([0,T), w - L^1((0,\infty), x\,dx))$ and, for each $T^* < T$, it also belongs to $L^\infty((0,T^*), L^1((0,\infty), dx))$;
ii) For all $t \in [0,T)$,

$$u(t) \coloneqq \rho - \int_0^\infty x f(t,x)\,dx > \Phi_0\,; \tag{1.4}$$

iii) For all $\varphi \in \mathcal{C}_c^1([0,T) \times [0,+\infty))$, there holds that

$$\int_0^T \int_0^\infty \left(\partial_t\varphi(t,x) + (a(x)u(t) - b(x))\partial_x\varphi(t,x)\right) f(t,x)\,dx\,dt \\ + \int_0^T \varphi(t,0)\mathfrak{n}(u(t))\,dt + \int_0^\infty \varphi(0,x) f^{\rm in}(x)\,dx = 0\,. \tag{1.5}$$

Remark. Recall that the weak topology on $L^1((0,\infty),(1+x)\,dx)$, denoted by the prefix w, is the topology induced by the dual space $L^\infty((0,\infty),(1+x)\,dx)$.

To construct a solution to the Lifshitz–Slyozov equation we will assume that the kinetic rates $\{a, b, \mathfrak{n}\}$ satisfy the following working hypotheses:

$$a,\, b \in \mathcal{C}^0([0,\infty)) \cap \mathcal{C}^1(0,\infty)\,, \tag{H1}$$

$$a' \text{ and } b' \text{ are bounded on } (1,\infty)\,, \tag{H2}$$

$$a(x) > 0 \text{ for all } x > 0 \text{ and } \tfrac{1}{a} \in L^1(0,1)\,, \tag{H3}$$

$$\Phi' \in L^1(0,1)\,, \tag{H4}$$

$$\mathfrak{n} \text{ is locally Lipschitz on } [\Phi_0,\infty)\,, \tag{H5}$$

where $\Phi = b/a$ is the ratio function. Moreover, we restrict the choice of initial data to

$$f^{\rm in} \in L^1((0,\infty),(1+x)\,dx)\,, \tag{H6}$$

$$u^{\rm in} \coloneqq \rho - \int_0^\infty x f^{\rm in} > \Phi_0\,, \tag{H7}$$

so that the balance of mass (1.4) makes sense at time $t = 0$ together with the regularity required on $f^{\rm in}$ in Definition 1.2.

Before we discuss these assumptions and state our existence result, let us introduced a Lemma that might help to interpret Definition 1.2 through the most common moment equations, which in turn will be useful for several estimates in the sequel.

Lemma 1.3 (Moment equations)**.** *Assume to be given the kinetic rates* $\{a, b, \mathfrak{n}\}$ *satisfying hypotheses* (H1)-(H2) *and* (H5)*, a constant* $\rho > \Phi_0$ *and a non-negative function* f^{in} *satisfying* (H6) *and* (H7)*. Let* $T > 0$ *and* f *be a solution to the Lifshitz-Slyozov equation in the sense of Definition 1.2 on* $[0, T)$ *with mass* ρ*, kinetic rates* $\{a, b, \mathfrak{n}\}$ *and initial value* f^{in}*. For all* $t \in [0, T)$ *and for every real-valued functions* h *defined on* $[0, \infty)$*, locally bounded such that* $h' \in L^\infty(0, \infty)$*, we have*

$$\int_0^\infty h(x) f(t,x)\,dx = \int_0^\infty h(x) f^{\mathrm{in}}(x)\,dx + \int_0^t \int_0^\infty (a(x)u(s) - b(x)) h'(x) f(s,x)\,dx\,ds + \int_0^t h(0)\mathfrak{n}(u(s))\,dt\,. \tag{1.6}$$

Moreover, f *belongs to* $L^\infty((0,T); L^1((0,\infty), dx))$*,* u *is continuously differentiable on* $(0, T)$ *and*

$$\frac{du(t)}{dt} = -u(t) \int_0^\infty a(x) f(t,x)\,dx + \int_0^\infty b(x) f(t,x)\,dx\,, \tag{1.7}$$

for all $t \in (0, T)$*.*

Proof. This proceeds via a standard regularization procedure. Plug $\varphi(t,x) = g(t)h(x)$ with $g \in \mathcal{C}_c^1((0,T))$ and $h \in \mathcal{C}([0,\infty))$ into Eq. (1.5) and observe that the distributional derivative of $\int_0^\infty h(x) f(t,x)\,dx$ belongs to $L^\infty(0,T)$ by Eq. (1.4), (H1), (H2) and the regularity point i) in Definition 1.2. Then the conclusion holds easily by identification of this derivative and since $f(0,x) = f^{\mathrm{in}}$ *a.e.* $x > 0$ by Eq. (1.5). We relax to h with bounded derivative by standard regularization and the fact that rates are sublinear (see (1.8) below) together with the regularity of f. Note that Eq. (1.7) is obtained by Eq. (1.6) with $h(x) = x$ and Eq. (1.4). □

Theorem 1.4 (Existence of solution)**.** *Assume to be given the kinetic rates* $\{a, b, \mathfrak{n}\}$ *satisfying hypotheses* (H1) *to* (H5)*, a constant* $\rho > \Phi_0$ *and a non-negative function* f^{in} *satisfying* (H6) *and* (H7)*. Then, there exists at least one solution to the Lifshitz–Slyozov equation in the sense of Definition 1.2 on* $[0, T)$ *with mass* ρ*, kinetic rates* $\{a, b, \mathfrak{n}\}$ *and initial value* f^{in}*, such that either* $T = \infty$ *or* $T < \infty$ *and* $u(t) \to \Phi_0$ *as* $t \to T$*. In particular, this solution* f *belongs to* $\mathcal{C}([0,T); w-L^1((0,\infty),(1+x)\,dx))$*, satisfies* $f(0,x) = f^{\mathrm{in}}(x)$ *for a.e.* $x > 0$ *and*

$$\lim_{x\to 0^+} (a(x)u(t) - b(x)) f(t,x) = \mathfrak{n}(u(t))\,,$$

for all $t \in (0,T)$*. This solution can be represented in terms of characteristics, cf. formula* (2.11) *below.*

Our set of running hypotheses entails the existence of a positive constant K_r such that

$$a(x) + b(x) \le K_r(1+x) \tag{1.8}$$

for all $x \ge 0$. Hypotheses (H1), (H2), (H3) and (H4) fit well with power law rates: $a(x) = a_0 x^\alpha$ and $b(x) = b_0 x^\beta$ for $x \ge 0$ in the relevant case $0 \le \alpha \le \beta \le 1$ with $a_0 > 0$, $b_0 \ge 0$ and $\alpha < 1$. Note that $\Phi(x) = \frac{b_0}{a_0} x^{\beta-\alpha}$ is such that Φ' is integrable at the origin and $\Phi_0 = b_0/a_0$ if $\alpha = \beta$, while $\Phi_0 = 0$ if $\alpha < \beta$. The case $\alpha > \beta$ is out of the scope of this paper since $\Phi_0 = \infty > \rho$ and then the flow is outgoing. Hypothesis (H5) is trivially satisfied for $\mathfrak{n}(z) = \mathfrak{n}_0 z^n$ for $z \geq 0$ with $n \geq 1$ and $\mathfrak{n}_0 \geq 0$; then for each ρ there exists a positive constant $K_{\mathfrak{n}}$ such that

$$|\mathfrak{n}(z_1) - \mathfrak{n}(z_2)| \leq K_{\mathfrak{n}} |z_1 - z_2| \tag{1.9}$$

for all $z \in (0, \rho)$. Condition (H6) on initial data seems to be optimal to make sense of the mass balance for the initial datum and to be able to account for the boundary in the formulation (1.5). Finally, hypothesis (H7) is essential so that we may consider inflow solutions.

Remark 1.5. In this paper we work with rates a and b having classical regularity on $(0, \infty)$; this can be relaxed to Lipschitz regularity. The actual difficulty in the analysis comes rather from the deterioration of the regularity at the origin (which includes the case of power law rates) combined with the boundary condition. In particular a' and b' need not be bounded around zero. The need of integrability of $1/a$ is related to the method of factorization of the flow we consider here and works well for power laws too. Indeed, we rewrite the flow as $a(x)u(t) - b(x) = a(x)(u(t) - \Phi(x))$ and we carry a partial integration of the characteristic (detailed in Sec. 2.1) to get a reparametrized flow of the form $u(t) - \Phi \circ A^{-1}(x)$, where A is the primitive of $1/a$. If $1/a$ is not integrable around zero, the return time of the characteristic towards the boundary is infinite, in which case no boundary condition is needed. We also mention that the integrability of Φ', which is equivalent to the integrability of $(\Phi \circ A^{-1})'$, is a standard assumption on the flow of a transport equation.

The solution constructed in Theorem 1.4 can be shown to be unique under some additional assumptions. First, we need some monotonicity of the function Φ around zero, namely

$$\text{There exists } x^* > 0 \text{ such that } \Phi \text{ is monotone on } [0, x^*). \tag{H8}$$

Moreover, we will need an extra moment hypothesis to control the tail of the solutions

$$f^{\text{in}} \in L^1((0, \infty), (1 + x + x^2)\, dx)\,. \tag{H9}$$

Theorem 1.6 (Uniqueness of solution)**.** *Under the hypotheses of Theorem 1.4, assume moreover* (H8) *to be true. Then, for any initial data satisfying assumption* (H9)*, and for all $T > 0$, there exists at most one solution to the Lifshitz–Slyozov equation on $(0, T)$ in the sense of Definition 1.2.*

Assumption (H8) is clearly satisfied for power laws and therefore it is not very restrictive in applications. Actually, we show in Section 3.2 below that Theorem 1.6 can be proved under slightly more general assumptions on the kinetic rates, see the assumptions (H8a)–(H8b) in that section. We believe that assumption (H9) is purely technical and related to the method of proof we used (inspired by [21]), but we have not been able to cope without it.

We finish this section by a theorem giving sufficient conditions for global solution to exist, as well as providing examples of maximal solutions defined in a finite time interval.

Theorem 1.7 (Global and local solutions)**.** *Under the hypotheses of Theorem 1.4, the following statements hold:*

• *Assume* $\Phi(x) \geq \Phi_0$ *for all* $x > 0$*. Then, the solution to the Lifshitz–Slyozov equation constructed in Theorem 1.4 is global, that is* $T = \infty$*.*

• *Assume that* f^{in} *is compactly supported, that* Φ *is convex and strictly decreasing and that there exists numbers* $\underline{a}$*,* $\overline{a}$ *such that* $0 < \underline{a} < a(x) < \overline{a} < \infty$ *for all* $x > 0$*. Then, the solution to the Lifshitz–Slyozov equation constructed in Theorem 1.4 is not global, that is,* u *reaches* Φ_0 *in finite time.*

Remark 1.8. First point covers the case of power law rates $b(x) = b_0 x^\beta$ and $a(x) = a_0 x^\alpha$ with $0 \leq \alpha \leq \beta < 1$. Note that when $\Phi_0 = 0$ we always have global existence.

In the proof of Theorem 1.7, we clearly show that in the second case, u reaches Φ_0 in finite time with a negative time derivative. Thus, were we able to extend smoothly this solution past the hitting time of Φ_0, it would have to be an outflow solution for some time interval, which calls for a broader concept of solution to the Lifshitz-Slyozov equation, which would unify inflow and outflow solutions. Note that the situation is completely symmetric, in the sense that the arguments given in Section 3.3 can be adapted to construct an outflow solution for which u stays below Φ_0 only on a finite time interval.

1.3. **Outline.** We prove important properties on characteristic curves associated to the linear inhomogeneous problem for Eq. (1.1)–(1.3) (when $t \mapsto u(t)$ is given) in Section 2. First, we study the characteristics in Section 2.1 and prove in Section 2.2 that they allow to transfer the value of the boundary condition and initial condition to the solution through a diffeomorphism. Then, these properties allow to define mild solutions in Section 2.3. In Section 3 we turn to the full non-linear problem. Existence of local-in-time solutions of the non-linear problem is proved in Section 3.1 thanks to a fixed point argument. In fact, we prove optimal prolongation of the solution up to $T = \infty$ or $T < \infty$ with $u(t) \to \Phi_0$ as $t \to T$. Then, uniqueness of solution is proved in Section 3.2 and finally criteria for global solutions are discussed in Section 3.3.

2. Characteristics curves and linear problem

All along this section we assume to be given $T > 0$, $\rho > 0$, $\{a, b, \mathfrak{n}\}$ kinetic rates, and u a function belonging to

$$\mathcal{BC}^+_\rho([0,T)) = \{u \mid u : [0,T) \to [0,\rho] \text{ continuous}\} \ .$$

We denote by

$$\overline{u}_T = \sup\{u(t) \mid t \in [0,T)\} \ , \text{ and } \underline{u}_T = \inf\{u(t) \mid t \in [0,T)\} \ .$$

We remark that, by definition, $0 \leq \underline{u}_T \leq \overline{u}_T \leq \rho$. Moreover, we assume that $\underline{u}_T > \Phi_0$ and also that assumptions (H1)–(H5) hold. We define

$$v(t,x) = a(x)u(t) - b(x)$$

for all $(t,x) \in \Omega_T$.

In the following two subsections we state a number of results concerning useful technical properties of the characteristic curves associated to the transport equation (1.1). These results are crucial to deal with solutions of the Lifshitz–Slyozov equation (1.1)–(1.3).

2.1. **Characteristics: definition and construction of the associated flow.** We first state a basic lemma on the characteristics curves associated to the transport equation (1.1).

Lemma 2.1. *For any $(t,x) \in \Omega_T$, there exists a unique maximal solution to*

$$\begin{aligned} \frac{dX(s;t,x)}{dt} &= v(s, X(s;t,x))\,, \\ X(t;t,x) &= x \end{aligned} \tag{2.1}$$

whose maximal interval is denoted by $J_{t,x}$. Moreover, the following properties hold true:

- *For any $(t_0,x_0) \in \Omega_T$ and $s_0 \in J_{t_0,x_0}$, there exists a neighborhood of (s_0,t_0,x_0) in $J_{t_0,x_0} \times \Omega_T$ such that $(s,t,x) \mapsto X(s;t,x)$ is well-defined and continuously differentiable;*
- *The semigroup property $X(t;s,X(s;t,x)) = x$ is satisfied for every $s \in J_{t,x}$;*
- *For every $s \in J_{t,x}$ we have*

$$\begin{aligned} \frac{dX(s;t,x)}{dx} &\coloneqq J(s;t,x) = \exp\left(- \int_s^t (\partial_x v)(\tau, X(\tau;t,x))\, d\tau \right)\,, \\ \frac{dX(s;t,x)}{dt} &= -v(t,x) J(s;t,x)\,; \end{aligned} \tag{2.2}$$

- *There exists a positive constant $C(T)$, independent of $u \in \mathcal{BC}^+_\rho([0,T))$, such that*

$$X(s;t,x) + \left| \frac{dX(s;t,x)}{dt} \right| \le C(T)(1+x) \tag{2.3}$$

for all (t,x) in Ω_T and s in $J_{t,x}$. As a consequence, each characteristic curve has a finite limit in $[0,\infty)$ at the end points of $J_{t,x}$.

Proof. Since v is continuous in the first variable and continuously differentiable in the second variable, for each (t,x) in Ω_T, there exist a unique solution $s \mapsto X(s;t,x)$ to system (2.1), called characteristic curve passing through x at time t. This solution is defined on a maximal interval $J_{t,x}$ in $[0,T)$, containing time t, with range in $(0,\infty)$. The semigroup property holds by the uniqueness of the maximal solution. Moreover, for each (t_0,x_0) in Ω_T and s_0 in J_{t_0,x_0}, there exists a neighborhood of (s_0,t_0,x_0) into $J_{t_0,x_0} \times \Omega_T$ such that the map $(s,t,x) \mapsto X(s;t,x)$ is well-defined and continuously differentiable, see for instance [36, Cap. II.3]. The derivatives (2.2) in t and x of $(s,t,x) \mapsto X(s;t,x)$ are classical. Thanks to Gronwall's lemma and the sublinearity of the rates (1.8), we also derive uniform bounds on characteristics curves to obtain (2.3), which prevents blow-up of the characteristic at the end points of $J_{t,x}$. □

In order to construct a solution to the Lifshitz–Slyozov equation (1.1) through the so-called characteristics formulation, we aim to know the life-time of these characteristics given by the lower and upper bounds of $J_{t,x}$. Particularly, we need to identify which characteristics go back to a positive x at time $s = 0$ and which ones go back to the boundary $x = 0$ in positive time $s > 0$. We can translate this problem as the study of the *enter-time* associated to the characteristic curve passing through x at time t -where (t,x) belongs to Ω_T, defined as the number

$$\sigma_t(x) \coloneqq \inf J_{t,x}\,.$$

The enter-time is the time at which the curve $s \mapsto (s, X(s;t,x))$ enters the phase-space Ω_T. Note that for $t = 0$, we readily have $\sigma_0(x) = 0$ for every $x > 0$.

Lemma 2.2. *Let (t,x) belonging to Ω_T^*. If $\sigma_t(x) = 0$ then $X(s;t,x) > 0$ for all s in $(0,t)$. Otherwise, if $\sigma_t(x) > 0$ then $\lim_{s\to\sigma_t(x)^+} X(s;t,x) = 0$.*

Proof. Since the solutions belong to $(0,\infty)$ the first statement readily follows from the definitions of $J_{t,x}$ and $\sigma_t(x)$. In the case $\sigma_t(x) > 0$, since a characteristic curve has a finite limit at the lower end of $J_{t,x}$, this limit is either positive or zero. But, if the limit is positive (say $\bar{x}$), thanks to the Cauchy-Lipschitz theory we can construct a prolongation of the characteristic curve in a neighborhood of $(\sigma_t(x), \bar{x})$, which contradicts the definition of $\sigma_t(x)$. Therefore the limit at $\sigma_t(x)^+$ vanishes. □

Due to the lack of Lipschitz regularity, the analysis of the characteristic curves will be tackled thanks to a reparametrization of the flow through a diffeomorphism, leading to a positive lower-bound of the time derivative of the new characteristic curves at the boundary $x = 0$. Thanks to assumption (H3), we define, for all $x > 0$,

$$A(x) := \int_0^x \frac{1}{a(y)}\,dy\,.$$

Clearly A is continuous differentiable on $(0,\infty)$ and (strictly) increasing. Moreover, since a is linearly bounded, $\lim_{x\to\infty} A(x) = +\infty$. Thus, A is one-to-one onto $(0,\infty)$; we can extend it continuously by taking $A(0) = 0$. The inverse A^{-1} is increasing and continuously differentiable on $(0,\infty)$ since $A^{-1\prime}(x) = a(A^{-1}(x))$ as $x > 0$. We set $A^{-1}(0) = 0$, which gives a continuous extension; the derivative may be also extended continuously by taking $A^{-1\prime}(0) = a(0)$. Moreover, $\Phi = b/a$ is continuously differentiable on $(0,\infty)$ by the assumptions on a and b. Hence $\Phi \circ A^{-1}$ is continuously differentiable on $(0,\infty)$. We define, for each (t,x) in Ω_T, the reparametrized transport field

$$V(t,x) = u(t) - \Phi \circ A^{-1}(x)\,.$$

Lemma 2.3. *For any $(t,y) \in \Omega_T$, there exists a unique maximal solution to*

$$\begin{cases} \dfrac{dB(s;t,y)}{dt} = V(s, B(s;t,y))\,, \\ B(t;t,y) = y \end{cases} \tag{2.4}$$

whose maximal interval is denoted by $\tilde{J}_{t,y}$. Moreover, the following properties hold true:

- *For any $(t_0,y_0) \in \Omega_T$ and $s_0 \in \tilde{J}_{t_0,y_0}$, there exists a neighbourhood of (s_0,t_0,y_0) in $\tilde{J}_{t_0,y_0} \times \Omega_T$ such that $(s,t,y) \mapsto B(s;t,y)$ is well-defined and continuously differentiable;*
- *The semigroup property $B(t;s,B(s;t,y)) = y$ is satisfied for every $s \in \tilde{J}_{t,y}$.*
- *For every $s \in \tilde{J}_{t,y}$ we have*

$$\begin{aligned} \frac{dB(s;t,y)}{dy} &:= I(s;t,y) = \exp\left(\int_s^t (a\cdot\Phi')(A^{-1}(B(\tau;t,y)))d\tau\right), \\ \frac{dB(s;t,y)}{dt} &= -V(t,y)I(s;t,y)\,; \end{aligned} \tag{2.5}$$

- *For any* $(t,x)\in\Omega_T$, *we have*

$$\tilde{J}_{t,A(x)} = J_{t,x} \text{ and } B(s;t,A(x)) = A(X(s;t,x))\,,\ \text{for any } s\in J_{t,x}\,. \tag{2.6}$$

Proof. Fix $(t,y)\in\Omega_T$. Since V is continuous and continuously differentiable in the second variable, there exists a unique solution $s\mapsto B(s;t,y)$ to system (2.4), defined on a maximal interval $\tilde{J}_{t,y}\subset[0,T)$ containing t and with range in $(0,\infty)$. Taking into account that

$$-\partial_y V(t,y) = (\Phi\circ A^{-1})'(y) = a(A^{-1}(y))\Phi'(A^{-1}(y))\,,\ \text{for all } y>0\,,$$

all the stated properties follow easily as in Lemma 2.1 (see e.g. [36, Cap. II.3]), except maybe (2.6). We now prove (2.6). Let $(t,x)\in\Omega_T$. First, $s\mapsto A(X(s;t,x))$ is a solution to system (2.4) with $A(X(t;t,x)) = A(x)$. Thus $J_{t,x}\subseteq\tilde{J}_{t,A(x)}$ and $B(s;t,A(x)) = A(X(s;t,x))$ for all $s\in J_{t,x}$. Define $Y(s;t,x) = A^{-1}(B(s;t,A(x)))$ for all $s\in\tilde{J}_{t,A(x)}$. Then Y is a solution to the original system (2.1) with $Y(t;t,x) = x$, thus $\tilde{J}_{t,A(x)}\subseteq J_{t,x}$. Therefore, $\tilde{J}_{t,A(x)} = J_{t,x}$ and (2.6) holds. □

Remark 2.4. We will repeatedly use in the proofs below that the derivatives of X and B with respect to their third argument are positive. In other words, uniqueness ensures that characteristics cannot cross and hence we have the following monotonicity property: given $x<y$, then for all $s\in J_{t,x}\cap J_{t,y}$ we have $X(s;t,x)<X(s;t,y)$ and $B(s;t,A(x))<B(s;t,A(y))$.

The control of the time derivative of X at the boundary $x=0$ is stated in the lemma below, thanks to the characteristics B.

Lemma 2.5. *There exists* $x_0>0$ *and* $\delta>0$ *(depending on* u *and* Φ*) such that, for all* $t\in[0,T)$ *and* $x\in(0,x_0)$,

$$u(t)-\Phi(x)\geq\delta\,. \tag{2.7}$$

Then, for every $(t,x)\in\Omega_T$ *and* $\tau\in J_{t,x}$ *such that* $X(\tau;t,x)<x_0$ *the following holds:*

- *the map* $s\mapsto X(s;t,x)$ *is increasing on* $(\sigma_t(x),\tau)$ *and for any* $s\in(\sigma_t(x),\tau)$, $X(s;t,x)<X(\tau;t,x)<x_0$;
- *for every* $s\in(\sigma_t(x),\tau)$ *we have the lower bound*

$$\frac{dA(X(s;t,x))}{dt} = \frac{dB(s;t,A(x))}{dt}\geq\delta\,.$$

Moreover, for all $(t,x)\in\Omega_T$,

- $J_{t,x} = (\sigma_t(x),T)$ *if* $t\in(0,T)$, *while* $J_{0,x}=[0,T)$, *and for every* $s\in[t,T)$ *we have*

$$X(s,t,x)\geq\min(x,x_0)\,.$$

Proof. The existence of a pair (x_0,δ) such that Eq. (2.7) holds is trivial from the continuity of Φ at zero and the running assumption $\underline{u}_T>\Phi_0$. Let $(t,x)\in\Omega_T$, we may rewrite the equation on the characteristic curves (2.1) as

$$\frac{dX(s;t,x)}{ds} = a(X(s;t,x))\,(u(t)-\Phi(X(s;t,x)))\,.$$

Since a is positive, the flow verifies $a(z)(u(t)-\Phi(z))>a(z)\delta>0$ for all $(t,z)\in[0,T)\times(0,x_0)$, which shows that the interval $(0,x_0)$ is negatively invariant. In other words, if there exists $\tau \in J_{t,x}$ such that $X(\tau;t,x) < x_0$ then, $X(s;t,x) < x_0$ and

$$\frac{dB(s;t,A(x))}{ds} = u(s) - \Phi \circ A^{-1}(B(s;t,A(x))) = u(s) - \Phi(X(s;t,x)) \geq \delta\,,$$

for all $s \in (\sigma_t(x), \tau)$, which proves the second point. Thus, first point directly follows from this fact and using that A is increasing with $B(s;t,A(x)) = A(X(s;t,x))$. We then prove last point. As $(0,x_0)$ is negatively invariant, and since the flow is positive on $(0,x_0)$, this also proves that $(0,x)$ is negatively invariant for each $x \in (0,x_0)$. We claim that (x,∞) is positively invariant for all $x \in (0,x_0]$. This can be proved arguing by contradiction: Let $y \in (x,\infty)$ and $t \in [0,T)$; if there exists $s > t$ such that $X(s;t,y) \leq x$, then for all times $\tau \leq s$, we have that $X(\tau,t,y) \leq x$ because $(0, X(s;t,y))$ is negatively invariant. We deduce that $X(t;t,y) = y \leq x$, which contradicts the premise and yields our claim. In fact, this argument readily entails $X(s;t,x) \geq \min(x_0,x)$ for all $(t,x) \in \Omega_T$ and $s \in J_{t,x} \cap (t,T)$. We conclude thanks to the lower bound and remarking that the Cauchy-Lipschitz theory allows to prolongate solutions up to time T by the regularity of the rates a and b -see Eq. (2.3). □

We end this section by the following technical lemma, which turns out to be crucial to bound the derivatives of B, see equations (2.5). This result also shows that assumption (H4), $\Phi' \in L^1(0,1)$, is close to be optimal to prevent concentration in finite time.

Lemma 2.6. *Let $\delta > 0$ and x_0 given by Lemma 2.5. For all $t \in (0,T)$, $(s,\tau,s_0) \in (0,t]^3$, $x_1 \in (0,x_0]$ and $x > 0$, if $\sigma_\tau(x) \leq s \leq s_0$ and $X(s_0;\tau,x) < x_1$, then*

$$\int_s^{s_0} |(a \cdot \Phi')(X(r;\tau,x))|\, dr \leq \frac{1}{\delta} \int_0^{x_1} |\Phi'(z)|\, dz.$$

Moreover, there exists a constant $C(T) > 0$ independent on t, s, s_0, τ, x_1 and x, such that for all $r \in (s_0,t)$,

$$X(s_0;\tau,x) \leq X(r;\tau,x) \leq C(T)(1+x). \tag{2.8}$$

Proof. By Lemma 2.5, for all $r \in (\sigma_\tau(x), s_0)$, we have $X(r;\tau,x) < X(s_0;\tau,x) < x_1 < x_0$ and $u(r) - \Phi(X(r;\tau,x)) \geq \delta$. Hence, using Eq. (2.1),

$$\begin{aligned}\int_s^{s_0} |(a \cdot \Phi')(X(r;\tau,x))|\, dr &= \int_s^{s_0} \left| \frac{\Phi'(X(r;\tau,x))}{u(r) - \Phi(X(r;\tau,x))} \frac{dX(r;\tau,x)}{dr} \right| dr \\ &\leq \frac{1}{\delta} \int_s^{s_0} \left| \Phi'(X(r;\tau,x)) \frac{dX(r;\tau,x)}{dr} \right| dr = \frac{1}{\delta} \int_{X(s;\tau,x)}^{X(s_0;\tau,x)} |\Phi'(z)|\, dz\,. \end{aligned} \tag{2.9}$$

We conclude as $0 < X(s;\tau,x) < X(s_0;\tau,x) < x_1$.

Now we prove (2.8). We deduce from the third point of Lemma 2.5 that, for all $r \in (s_0,T)$, we have $X(r;\tau,x) = X(r;s_0,X(s_0;\tau,x)) \geq X(s_0;\tau,x)$. Then, using the bound in Eq. (2.3), we obtain that $X(r;\tau,x) \leq C(T)(1+x)$ for some constant $C(T) > 0$, which concludes the proof. □

2.2. **Diffeomorphism through the characteristic curves.** In this section we give rigorous sense to the concept of characteristics curves starting from $x = 0$ at a positive time; this cannot be achieved directly from system (2.1) due to the lack of derivative at the origin. Nevertheless, the analysis of the map $x \mapsto \sigma_t(x)$ at each time t allows us to single out a unique characteristic curve starting from $x = 0$ at time $s \in (0,t)$, in fact the one being at some x satisfying $s = \sigma_t(x)$. This provides an interpretation of $X(t;s,0)$ being the inverse of $\sigma_t(x)$, that is $X(t;s,0) = \sigma_t^{-1}(s)$. These considerations are intricately related with the fact that $a(x)$ is the driving term at $x = 0$ in the differential equation (2.1) whenever $\underline{u}_T > \Phi_0$. Namely,

$$v(t,x) = a(x)(u(t) - \Phi(x)) = a(x)[u(t) - \Phi_0 + (\Phi_0 - \Phi(x))]$$

but $\Phi_0 - \Phi(x)$ has little influence when x is close to the origin. Then an integrability condition for $1/a$ at the origin, by assumption (H3), arises naturally -see e.g. [10]. The methodology of the characteristics is classical; however, most proofs are quite technical due to the lack of derivative at the origin, and depend on the change of variable procedure presented in subsection 2.1. For the reader's convenience, we shall defer some proofs to the Annex in Section 4.

By Lemma 2.2 we know that when $\sigma_t(x) > 0$ the characteristic curve reaches the axis $x = 0$ at time $\sigma_t(x)$. Moreover, uniqueness to system (2.1) yields that the family of characteristic curves is a totally ordered family; therefore, we may tell whether characteristic curves came back from zero or not in terms of the separating point

$$x_c(t) := \inf \{x > 0 \mid \sigma_t(x) = 0\}$$

defined for each t in $[0,T)$.

Lemma 2.7. *For each $t \in (0,T)$ we have the following properties:*

- *the value $x_c(t)$ is finite and positive,*
- *σ_t is a non-increasing map which is positive on $(0, x_c(t))$,*
- *σ_t vanishes on $(x_c(t), \infty)$.*

Moreover, for $t = 0$ we have: $x_c(0) = 0$ and σ_0 is constantly equal to zero.

In fact $t \mapsto x_c(t)$ can be interpreted as the characteristic curve starting from $x = 0$ at time zero, see Proposition 2.8 below. Also, note that the characteristic curves $s \mapsto X(s;t,x)$ do not leave the phase-space Ω_T for times in (t,T), justifying the terminology of "inflow". Below we state the two main propositions that justify the use of a characteristic formulation for the Lifshitz–Slyozov equation (1.1).

Proposition 2.8. *For each $t \in (0,T)$, the map $x \mapsto X(t;0,x)$ is an increasing $\mathcal{C}^1$-diffeomorphism from $(0,\infty)$ to $(x_c(t),\infty)$ with derivative given by $J(t;0,x)$ in Eq. (2.2). Moreover, we have that $\lim_{x\to 0^+} X(t;0,x) = x_c(t)$.*

Proposition 2.9. *For each $t \in (0,T)$, the map $s \mapsto \sigma_t^{-1}(s)$ is a decreasing $\mathcal{C}^1$-diffeomorphism from $(0,t)$ to $(0,x_c(t))$ satisfying, for some constant $C(T)$ independent of the given $u \in \mathcal{BC}^+_\rho$,*

$$\sigma_t^{-1}(s) \le C(T)\,.$$

Its derivative is given by

$$\frac{d\sigma_t^{-1}(s)}{dt} = -a(\sigma_t^{-1}(s))(u(s) - \Phi_0)\exp\left(-\int_s^t (a\cdot\Phi')(\sigma_\tau^{-1}(s))\,d\tau\right) \tag{2.10}$$

for all $t \in (0,T)$ and $s \in (0,t)$. Moreover, $\sigma_t^{-1}(s) = \lim_{x\to 0^+} X(t;s,x)$ and $\sigma_\tau^{-1}(\sigma_t(x)) = X(\tau;t,x)$ for all $(t,x) \in \Omega_T^*$ and $\tau \in J_{t,x}$.

The technical proofs of Lemma 2.7 and these two Propositions 2.8 and 2.9 are postponed to the Annex in Section 4.

2.3. **Mild formulation and regularities.** In this section we assume the running hypotheses of Section 2, that is we are given a function $u \in \mathcal{BC}^+_\rho([0,T))$ such that $\underline{u}_T > \Phi_0$ and $\{a, b, \mathfrak{n}\}$ admissible kinetic rates satisfying assumptions (H1)–(H5). We also fix $f^{\rm in}$, a non-negative function on $(0,\infty)$ satisfying (H6). Thanks to Propositions 2.8 and 2.9, we define for a.e. $(t,x) \in \Omega^*_T$

$$f(t,x) = f^{\rm in}(X(0;t,x))J(0;t,x)\mathbf{1}_{(x_c(t),\infty)}(x) + \mathfrak{n}(u(\sigma_t(x))|\sigma_t'(x)|\mathbf{1}_{(0,x_c(t))}(x)\,, \tag{2.11}$$

where $\mathbf{1}_I$, stands for the indicator function of an interval I. Indeed, $\sigma_t'(x) = 1/(\sigma_t^{-1\prime}(\sigma_t(x))$ is defined for all $t \in (0,T)$ and $x \in (0, x_c(t))$.

Remark 2.10. Note that (2.11) makes sense even if $f^{\rm in}$ is only defined almost everywhere. This is due to the fact that for every null set $\mathcal{O}$, the set $\mathcal{O}_t := \{x \mid X(0;t,x) \in \mathcal{O}\} = X(t;0,\mathcal{O})$ is also a null set, being the image of a null set by a $\mathcal{C}^1$-diffeomorphism. Indeed, we have that

$$\int_0^T \int_0^\infty \mathbf{1}_{\mathcal{O}_t}(x)\,dx = \int_{\mathcal{O}} \left(\int_0^T J(0;t,y)\,dt \right) dx = 0$$

via change of variables and Fubini's theorem. Hence $\{(t,x) \mid t \in (0,T),\, x \in \mathcal{O}_t\}$ is a null set and thus f is defined a.e. in Ω^*_T.

The main result of the section is the following moment formulation.

Proposition 2.11. *The function f defined in Eq.* (2.11) *belongs to the functional space $\mathcal{C}\left([0,T); w - L^1((0,\infty),(1+x)dx)\right)$. It satisfies*

$$\int_0^\infty h(x)f(t,x)\,dx = \int_0^\infty h(x)f^{\rm in}(x)\,dx + h(0)\int_0^t \mathfrak{n}(u(s))\,ds + \int_0^t\int_0^\infty (a(x)u(s) - b(x))h'(x)f(s,x)\,dx\,ds\,, \tag{2.12}$$

for all $t \in (0,T)$ and any real-valued function h defined on $[0,\infty)$, locally bounded such that $h' \in L^\infty(0,\infty)$. Moreover, there holds that

$$\lim_{x\to 0}(a(x)u(t) - b(x))f(t,x) = \mathfrak{n}(u(t))\,. \tag{2.13}$$

We split the proof of this result into a number of intermediate statements.

Lemma 2.12. *The family $\{f(t,\cdot) \mid t \in (0,T)\}$ constructed via Eq.* (2.11) *is weakly relatively compact in $L^1((0,\infty),(1+x)dx)$. In particular,*

$$\sup_{t\in[0,T)} \int_0^\infty (1+x)f(t,x)\,dx < \infty. \tag{2.14}$$

Proof of lemma 2.12. The result will follow as a consequence of Dunford-Pettis' theorem. Since f is non-negative, we are to prove the following:

i) Bound (2.14),

ii) $\displaystyle\lim_{n\to+\infty} \sup_{t\in[0,T]} \int_n^\infty f(t,x)(1+x)\,dx = 0,$

iii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{t\in[0,T]} \int_E f(t,x)(1+x)\,dx < \varepsilon$$

for every Lebesgue measurable set E with measure $|E| < \delta$.

Point i. We integrate in (2.11) and use the diffeomorphisms in Propositions 2.8 and 2.9 to obtain

$$\int_0^\infty f(t,x)\,dx = \int_0^\infty f^{\mathrm{in}}(x)\,dx + \int_0^t \mathfrak{n}(u(s))\,ds,$$

for each $t \in (0,T)$. This is bounded for each t since $\mathfrak{n}$ is bounded on $[\Phi_0,\rho]$, f^{in} belongs to $L^1((0,\infty),(1+x)dx)$ and u is continuous with range in $[\Phi_0,\rho]$. In a similar way, using the bound (2.3) and the bound in Proposition 2.9 we have that for each $t \in (0,T)$,

$$\int_0^\infty xf(t,x)\,dx = \int_0^\infty X(t;0,x)f^{\mathrm{in}}(x)\,dx + \int_0^t \sigma_t^{-1}(s)\mathfrak{n}(u(s))\,ds,$$

is bounded. This ends the proof.

Point ii. Note first that there exists a constant $C(T)>0$ such that $x_c(t) \le C(T)$ for all $t \in (0,T)$; this follows from bound (2.3) and the limit in Proposition 2.8. Choose N large enough such that $N \ge x_c(t)$ for all $t \in (0,T)$. Then, integrating (2.11) and changing variables we obtain that

$$\int_n^\infty (1+x)f(t,x)\,dx = \int_{X(0;t,n)}^\infty (1+X(t,0,x))f^{\mathrm{in}}(x)\,dx$$

for all $t \in (0,T)$ and $n \ge N$. Since $X(t,0,x) \le C(T)(1+x)$ again from bound (2.3), we have that

$$\int_n^\infty (1+x)f(t,x)\,dx \le C(T)\int_{X(0;t,n)}^\infty (1+x)f^{\mathrm{in}}(x)\,dx$$

increasing the value of the constant $C(T)$ if needed. Next we notice that $n \le C(T)(1+X(0;t,n))$ after (2.3) and the semigroup property. Hence, thanks to integrability of f^{in}, we can pass to the limit $n \to \infty$, uniformly in t, to obtain the desired property.

Point iii. Let E be a Lebesgue measurable set. We estimate the integrals over $E \cap (0,x_c(t))$ and $E \cap (x_c(t),\infty)$ separately. Thanks to Eqs. (2.2) and (2.3), we have

$$\begin{aligned}\int_{E\cap(x_c(t),\infty)} (1+x)f(t,x)\,dx &= \int_{E\cap(x_c(t),\infty)} (1+x)f^{\mathrm{in}}(X(0;t,x))J(0;t,x)\,dx \\ &\le C(T)\int_{X(0;t,E\cap(x_c(t),\infty))} (1+x)f^{\mathrm{in}}(x)\,dx \qquad (2.15)\end{aligned}$$

for some constant $C(T)>0$ independent of time $t \in [0,T)$. Let x_0 be given by Lemma 2.5 and let $\bar{x}$ be such that $X(s;0,x_0) \le \bar{x}$ for all $s \in [0,T)$ -this is possible thanks to Eq. (2.3). Note that for all $s,t \in [0,T)$ and $x > x_c(t)$ we have

$$X(s;t,x) > X(s;t,\bar{x}) > X(s;t,X(t;0,x_0)) = X(s;0,x_0) > x_0.$$

This is due to the monotonicity (Remark 2.4), the semigroup property and invariance. Now we estimate the measure of $X(0;t,E\cap(x_c(t),\infty))$ for $t \in [0,T)$ as follows:

$$|X(0;t,E\cap(x_c(t),\infty))| = \int_{E\cap(x_c(t),\infty)} J(0;t,x)\,dx$$
$$\leq \int_{E\cap(x_c(t),\bar{x})} J(0;t,x)\,dx + |E\cap(\bar{x},\infty)| \exp\left(T(\|a'\|_{L^\infty(x_0,\infty)}\rho + \|b'\|_{L^\infty(x_0,\infty)})\right). \tag{2.16}$$

Here we used Eq. (2.2) and assumptions (H1)-(H2). Now we proceed to bound the Jacobian. Since $A(X(0;t,x)) = B(0;t,A(x))$ using the derivatives in the third variable for X and B we get

$$J(0;t,x) = \frac{a(X(0;t,x))}{a(x)} I(0;t,A(x))$$

for all $x > x_c(t)$. To proceed further we use Eq. (2.3) to fix x^* such that $X(0;t,x) \leq x^*$ for all $x \in (x_c(t),\bar{x})$. By Eq. (2.16) above, we obtain

$$|X(0;t,E\cap(x_c(t),\infty))| \leq \|a\|_{L^\infty(0,x^*)} \int_{E\cap(x_c(t),\bar{x})} \frac{1}{a(x)} I(0;t,A(x))\,dx + C(T)|E|\,. \tag{2.17}$$

We now bound I. Given $x \in (x_c(t),\bar{x})$, we either have $X(0;t,x) < x_0$ or $X(0;t,x) \geq x_0$; we discuss both cases in turn. On one hand, if $X(0;t,x) \geq x_0$ we use Lemma 2.5 to deduce that for all $s \in (0,T)$, $x_0 \leq X(s;t,x) \leq x^*$. Recall that I is defined in Eq. (2.5). Thus, noticing that $a\Phi'$ is continuous on $(0,\infty)$ by (H1) and (H3),

$$I(0;t,A(x)) \leq \exp\left(T\|a\Phi'\|_{L^\infty(x_0,x^*)}\right). \tag{2.18}$$

On the other hand, if $X(0;t,x) < x_0$, there exists s_0 such that $X(s_0;t,x) = x_0$ and then $X(s;t,x) < x_0$ for all $s \in (0,s_0)$. So, by Lemma 2.6

$$|I(0;t,A(x))| \leq \exp\left(\tfrac{1}{\delta}\int_0^{x_0} |\Phi'(z)|\,dz + T\|a\phi'\|_{L^\infty(x_0,x^*)}\right), \tag{2.19}$$

where $\delta > 0$ is given in Lemma 2.5 together with x_0. In conclusion, combining Eqs. (2.17), (2.18) and (2.19) we obtain

$$|X(0;t,E\cap(x_c(t),\infty))| \leq C(T)\left(\int_{E\cap(0,\bar{x})} \frac{1}{a(x)}\,dx + |E|\right).$$

Given that $1/a \in L^1(0,1)$ and $f^{\text{in}}(x)$ is integrable, Eq. (2.15) entails

$$\lim_{|E|\to 0} \sup_{t\in[0,T]} \int_{E\cap(x_c(t),\infty)} (1+x) f(t,x)\,dx = 0\,. \tag{2.20}$$

It remains to do the same with

$$\int_{E\cap(0,x_c(t))} (1+x) f(t,x)\,dx = \int_{E\cap(0,x_c(t))} (1+x)\mathfrak{n}(u(\sigma_t(x))|\sigma_t'(x)|\,dx\,.$$

Recall that $\mathfrak{n}$ is bounded on $[\Phi_0,\rho]$ and u takes values in that interval; recall also that $x_c(t)$ is uniformly bounded on $(0,T)$. Therefore, there is some $C(T)$ such that

$$\int_{E\cap(0,x_c(t))} (1+x) f(t,x)\,dx \leq C(T) \int_{E\cap(0,x_c(t))} |\sigma_t'(x)|\,dx. \tag{2.21}$$

We now consider this last integral. We observe that for all $x \in (0, x_c(t))$

$$\sigma_t'(x) = \frac{1}{{\sigma_t^{-1}}'(\sigma_t(x))} = -\frac{1}{a(x)(u(\sigma_t(x)) - \Phi_0)} \exp\left(\int_{\sigma_t(x)}^t (a \cdot \Phi')(X(\tau; t, x))\, d\tau\right), \tag{2.22}$$

where we used that $X(\tau; t, x) = \sigma_\tau^{-1}(\sigma_t(x))$ by Proposition 2.9. Thanks to Lemmas 2.2 and 2.7 we have that $\lim_{s \to \sigma_t(x)} X(s; t, x) = 0$ whenever $x \in (0, x_c(t))$. Thus, for each $(t, x) \in (0, T) \times (0, x_c(t))$, there exists $s_0 \in (\sigma_t(x), t]$ such that $X(s; t, x) < x_0$ for all $s \in (\sigma_t(x), s_0)$ where x_0 is given by Lemma 2.5. Using Lemma 2.6,

$$\int_{\sigma_t(x)}^t (a \cdot \Phi')(X(\tau; t, x))\, d\tau \leq \frac{1}{\delta} \int_0^{x_0} |\Phi'(z)|\, dz + T \|a\Phi'\|_{L^\infty(x_0, C(T))}\,.$$

Here $C(T) > 0$ is some constant which bounds $X(s; t, x)$ uniformly in $s, t \in (0, T)$ and $x \in (0, x_c(t))$ -see Lemma 2.1. Finally, since $u(s) - \Phi_0 > \delta$, we have

$$|\sigma_t'(x)| \leq \frac{C(T)}{\delta a(x)}$$

for all $s \in (0, T)$, again by Lemma 2.5. By assumption (H3) the right hand side of the last estimate is integrable around the origin and hence, by Eq. (2.21),

$$\lim_{|E| \to 0} \int_{E \cap (0, x_c(t))} (1 + x) f(t, x)\, dx \leq \frac{C(T)}{\delta} \lim_{|E| \to 0} \int_{E \cap (0, x_c(t))} \frac{1}{a(x)}\, dx = 0. \tag{2.23}$$

Combining limits (2.20) and (2.23) finishes the proof. □

Lemma 2.13. *The function f in Eq.* (2.11) *satisfies*

$$\int_0^T \int_0^\infty (\partial_t \varphi(t, x) + (a(x)u(t) - b(x))\partial_x \varphi(t, x)) f(t, x)\, dx\, dt + \int_0^\infty \varphi(0, x) f^{\mathrm{in}}(x)\, dx + \int_0^T \varphi(t, 0)\mathfrak{n}(u(t))\, dt = 0 \tag{2.24}$$

for all $\varphi \in \mathcal{C}_c^1([0, T) \times [0, +\infty))$.

Proof. Let $\varphi \in \mathcal{C}_c^1([0, T) \times [0, +\infty))$, and define

$$\psi(t, x) = -(\partial_t \varphi(t, x) + (a(x)u(t) - b(x))\partial_x \varphi(t, x)), \quad (t, x) \in \Omega_T\,. \tag{2.25}$$

Using the definition of f in Eq. (2.11), its integrability in Lemma 2.12 and equation (2.25), we obtain

$$\begin{aligned}\int_0^T \int_0^\infty (\partial_t \varphi(t, x) &+ (a(x)u(t) - b(x))\partial_x \varphi(t, x)) f(t, x)\, dx\, dt \\ &= -\int_0^T \int_{x_c(t)}^\infty \psi(t, x) f^{\mathrm{in}}(X(0; t, x)) J(0; t, x)\, dx\, dt \\ &\quad - \int_0^T \int_0^{x_c(t)} \psi(t, x)\mathfrak{n}(u(\sigma_t(x)))|\sigma_t(x)'|\, dx\, dt\,. \end{aligned} \tag{2.26}$$

Using the changes of variables in Propositions 2.8 and 2.9 and Fubini's theorem, we have

$$\int_0^T \int_{x_c(t)}^\infty \psi(t,x) f^{\rm in}(X(0;t,x)) J(0;t,x)\, dx\, dt \\ = \int_0^\infty \left(\int_0^T \psi(t, X(t;0,x)) \right) f^{\rm in}(x)\, dx\, dt\,, \quad (2.27)$$

and

$$\int_0^T \int_0^{x_c(t)} \psi(t,x) \mathfrak{n}(u(\sigma_t(x))) |\sigma_t(x)'|\, dx\, dt \\ = \int_0^T \left(\int_s^T \psi(t, \sigma_t^{-1}(s))\, dt \right) \mathfrak{n}(u(s))\, ds\,. \quad (2.28)$$

By definition of the characteristics curves (2.1) and using the definition of ψ in Eq. (2.25), we have

$$\frac{d}{ds}[\varphi(s, X(s;t,x))] = -\psi(s, X(s;t,x)) \quad (2.29)$$

for all $(t,x) \in \Omega_T$ and $s \in (\sigma_t(x), T)$. We stress that this equation remains true for $t = 0$ since, by Lemma 2.2, $X(s;0,x) > 0$ for all $s > 0$. Hence, integrating Eq. (2.29) over $(0,T)$ and since $\varphi(T,x) = 0$ for all $x > 0$, this yields

$$\varphi(0,x) = \int_0^T \psi(t; X(t;0,x))\, dt$$

for $x > 0$. We can insert this relation into equation (2.27) to obtain

$$\int_0^T \int_{x_c(t)}^\infty \psi(t,x) f^{\rm in}(X(0;t,x)) J(0;t,x)\, dx\, dt = \int_0^\infty \varphi(0,x) f^{\rm in}(x)\, dx\, dt\,. \quad (2.30)$$

Finally, by Proposition 2.9, we have $\psi(t, \sigma_t^{-1}(s)) = \lim_{x\to 0} \psi(t, X(t;s,x))$. Thus, using the dominated convergence theorem and equation (2.29),

$$\int_s^T \psi(t, \sigma_t^{-1}(s))\, dt = \lim_{x\to 0} \int_s^T \psi(t, X(t;s,x))\, dt = \varphi(t,0)$$

for all $t \in (0,T)$. Replacing this last relation in Eq. (2.28) we obtain

$$\int_0^T \int_0^{x_c(t)} \psi(t,x) \mathfrak{n}(u(\sigma_t(x))) |\sigma_t(x)'|\, dx\, dt = \int_0^T \varphi(t,0) \mathfrak{n}(u(t))\, dt\,. \quad (2.31)$$

Inserting Eqs. (2.30) and (2.31) into Eq. (2.26) ends the proof. □

Proof of Proposition 2.11. We can show that Eq. (2.24) is satisfied by f whenever $\varphi(t,x) = g(t)h(x)$, with $g \in \mathcal{C}_c^1(0,T)$ and $h \in \mathcal{C}_c^0([0,\infty))$ with $h' \in L^\infty(0,\infty)$. This follows from a standard regularization argument, together with the fact that f belongs to $L^\infty((0,T); L^1(0,\infty))$, Eq. (2.14), and the fact that the rates are locally bounded. Then, again by regularization, Eq. (2.24) is shown to be true for h locally bounded such that $h' \in L^\infty(0,\infty)$, namely,

$$\int_0^T g'(t)\int_0^\infty h(x)f(t,x)\,dx + \int_0^T g(t)\int_0^\infty (a(x)u(t)-b(x))h'(x)f(t,x)\,dx\,dt \\ + h(0)\int_0^T g(t)\mathfrak{n}(u(t))\,dt = 0\,.$$

Here we used that f belongs to $L^\infty((0,T);L^1((0,\infty),(1+x)dx))$ and the sublinearity of the rates; note that h has a well-defined limit at the origin. This entails that the map $t\mapsto \int_0^\infty h(x)f(t,x)\,dx$ has a bounded time derivative, which yields (2.12). We have in particular that $t\mapsto\int_0^\infty(1+x)h(x)f(t,x)\,dx$ is continuous for all $h\in\mathcal{C}_c^0(0,\infty)$, which is improved up to $h\in L^\infty(0,\infty)$ thanks to Lemma 2.12 and implies the claimed regularity of f. To finish the proof we analyze the limit (2.13). Let $t\in(0,T)$, we have

$$f(t,x) = \mathfrak{n}(u(\sigma_t(x)))|\sigma_t'(x)| \quad \text{a.e. } x\in(0,x_c(t))\,.$$

The right hand side being continuous in x we may choose a version of f that is continuous on $(0,x_c(t))$. Then from Eq. (2.22)

$$(a(x)u(t)-b(x))f(t,x) = \mathfrak{n}((u(\sigma_t(x)))\frac{u(t)-\Phi(x)}{|u(\sigma_t(x))-\Phi_0|}e^{\left(\int_{\sigma_t(x)}^t (a\cdot\Phi')(X(\tau;t,x))\,d\tau\right)}\,.$$

Thanks to Proposition 2.9, the factor in front of the exponential converges to $\mathfrak{n}(u(t))$ as $x\to 0^+$. It remains to prove that

$$\lim_{x\to 0^+}\int_{\sigma_t(x)}^t (a\cdot\Phi')(X(\tau;t,x))\,d\tau = 0\,.$$

Consider x_0 and δ given by Lemma 2.5 and let $x<x_0$ so that for all $\tau\in(\sigma_t(x),t)$ we have $X(\tau;t,x)<x_0$. Then, by Lemma 2.6,

$$\int_{\sigma_t(x)}^t |(a\cdot\Phi')(X(\tau;t,x))|\,d\tau \le \frac{1}{\delta}\int_0^x |\Phi'(z)|\,dz\,.$$

This last term vanishes as $x\to 0$, which concludes the proof. □

3. The non-linear problem

3.1. **Existence of solutions.** We follow the lines of [6] to show existence of local-in-time inflow solutions via the Schauder fixed point theorem. All along this section we assume to be given $T>0$, $\rho>0$ and $\{a,b,\mathfrak{n}\}$ admissible kinetic rates. Moreover, we assume that $\Phi_0<\rho$ and we let $f^{\text{in}}\in L^1((0,\infty),(1+x)\,dx)$ be non-negative and such that

$$u^{\text{in}} := \rho - \int_0^\infty x f^{\text{in}}(x)\,dx > \Phi_0\,.$$

Let $\delta>0$ such that $2\delta < u^{\text{in}}-\Phi_0$, and define

$$\mathcal{B}_\delta([0,T)) = \left\{u\in\mathcal{BC}_\rho^+([0,T)) \;\middle|\; u(0)=u^{\text{in}} \text{ and } \Phi_0+\delta\le u(t)\le\rho,\ \forall t\in[0,T)\right\}\,.$$

For each $u\in\mathcal{B}_\delta([0,T))$, we can define the density f given by Eq. (2.11) and then the function

$$v(t) = G(u)(t) = \left(\rho - \int_0^\infty x f(t,x)\,dx\right)\vee(\Phi_0+\delta)$$

for all $t \in [0,T)$ where $x \vee y$ denotes the maximum between x and y in $\mathbb{R}$. Our aim in this section is to prove the existence of a fixed point for the operator $u \mapsto G(u)$. We observe by construction that $\Phi_0 + \delta \le v(t) \le \rho$ and, by Proposition 2.11, v has a derivative in the sense of distributions belonging to $L^\infty(0,T)$ which is given for *a.e.* $t \in (0,T)$ by

$$v'(t) = \begin{cases} -\dfrac{d}{dt}\displaystyle\int_0^\infty xf(t,x)\,dx, & \text{if } \rho - \int_0^\infty xf(t,x)\,dx \ge \Phi_0 + \delta \\ 0, & \text{otherwise.} \end{cases}$$

Hence v is continuous, and thus G is a map from $\mathcal{B}_\delta([0,T))$ into itself. Moreover, it directly follows from Proposition 2.11 and the estimate in Lemma 2.12 that the derivative v' is uniformly bounded on $(0,T)$, independently on u. Thus, the image of $\mathcal{B}_\delta([0,T))$ is compact for the uniform topology. The remainder of this section is to prove the continuity of the operator G and then Theorem 1.4.

In the sequel, for a given sequence $\{u^n\}$ in $\mathcal{BC}_\rho^+$, we denote by X^n the solution to Eq. (2.1) associated to u^n and by $\sigma_t^{-1,n}$ the inverse function of the enter-time σ_t^n associated to X^n.

Lemma 3.1. *Let $\{u^n\}$ be a sequence in $\mathcal{B}_\delta([0,T))$ converging (uniformly) to u. For each $x > 0$, $X^n(\cdot;0,x)$ converges uniformly to $X(\cdot;0,x)$ on $[0,T)$ as $n \to \infty$.*

Proof. Fix $x > 0$. Thanks to the bounds in Eq. (2.3) and the continuity in the second variable of X^n,

$$|X^n(t;0,x)| \le C(T)(1+x)$$

for all $t \in (0,T)$ with some constant $C(T) > 0$ independent on n. Moreover,

$$\left|\frac{d}{dt}X^n(t;0,x)\right| \le K(\rho+1)T$$

where K can be taken as the maximum of a and b on the interval $[0, C(T)(1+x)]$. Thus the sequence $X^n(\cdot;0,x)$ is relatively compact and up to a subsequence, converges to a continuous function Y on $[0,T)$. Inspecting the equation on X^n we realize that the limit satisfies

$$Y(t) = x + \int_0^t (a(Y(s))u(s) - b(Y(s)))\,ds\,.$$

Thus Y is the unique solution to Eq. (2.1) with $Y(0) = x$ and therefore it coincides with the characteristic curve $s \mapsto X(s;0,x)$ associated to u. By uniqueness of the limit the full sequence converges and the result follows. □

Lemma 3.2. *Let $\{u^n\}$ be a sequence in $\mathcal{B}_\delta([0,T))$ converging (uniformly) to u. For each $t \in (0,T)$, $\sigma_t^{-1,n}$ converges pointwise to σ_t^{-1} as $n \to \infty$.*

Proof. Let $t \in (0,T)$ and $s \in (0,t)$. Define $x^n = \sigma_t^{-1,n}(s)$ for each $n \ge 1$. By Proposition 2.9 the sequence $\{x^n\}$ is bounded; denote by $\bar{x}$ this bound. Consider a subsequence of $\{x^n\}$ (not relabelled) which converges to some x. Thanks to Eq. (2.3) there is a constant $C(T)$ such that for all $\tau \in (s,t)$ and $n \ge 1$,

$$X^n(\tau;t,x^n) \le x^* := C(T)(1+\bar{x})\,.$$

Then, by Eq. (2.1),

$$\left|\frac{dX^n(\tau;t,x^n)}{d\tau}\right| \le \sup_{x\in(0,x^*)} (|a(x)|\rho + |b(x)|).$$

Hence, up to a subsequence, the functions $\tau \mapsto X^n(\tau; t, x^n)$ converge uniformly on $[s,t]$ to a continuous function, which we denote by $\tau \mapsto Y(\tau)$. Moreover, for all $n \geq 1$ and $\tau \in [s,t]$ we have

$$x^n - X^n(\tau; t, x^n) = \int_\tau^t [a(X^n(r;t,x^n))u^n(r) - b(X^n(r;t;x^n))]\, dr,$$

and at the limit $n \to \infty$,

$$Y(\tau) = x - \int_\tau^t [a(Y(r))u(r) - b(Y(\tau))]\, dr\,.$$

We observe that Y solves Eq. (2.1) with initial data $Y(t) = x$ on $[s,t]$, so $Y(\tau) = X(\tau; t, x)$ by uniqueness and in particular $\sigma_t(x) \leq s$. Finally, since $X^n(s;t,x^n) = 0$ for all $n \geq 1$, we have at the limit that $Y(s) = X(s;t,x) = 0$ and so $\sigma_t(x) = s$. In conclusion, from any subsequence of $\sigma_t^{-1,n}(s)$ we can extract a subsequence converging to $\sigma_t^{-1}(s)$, so the full sequence converges. □

We are now ready to prove the continuity of G.

Proposition 3.3. *Let $T > 0$. The operator G is continuous on $\mathcal{B}_\delta([0,T))$.*

Proof. Let u^n be a sequence in $\mathcal{B}_\delta([0,T))$ converging uniformly to u. Let f^n be the function associated to u^n that is given by Eq. (2.11). Thus

$$\int_0^\infty x f^n(t,x)\, dx = \int_0^\infty X^n(t,0,x) f^{\mathrm{in}}(x)\, dx + \int_0^t \sigma_t^{n,-1}(s)\mathfrak{n}(u^n(s))\, ds\,,$$

for all $t \in (0,T)$. Combining Lemmas 3.1 and 3.2 with the bounds in Eq. (2.3) and Proposition 2.9, we can use the dominated convergence theorem to show that

$$\int_0^\infty x f^n(t,x)\, dx \to \int_0^\infty x f(t,x)\, dx$$

for all $t \in (0,T)$, where f is the function associated to u that is given by Eq. (2.11). Thus $v^n(t) = G(u^n)(t)$ converges to $v(t) = G(u)(t)$ for all $t \in (0,T)$. Since the derivatives of v^n are uniformly bounded in $L^\infty(0,T)$, as mentioned above, the convergence is uniform. □

Proof of Theorem 1.4. The previous developments in this section enable us to apply Schauder's fixed point theorem. Thus, there exists a fixed point to G, which means that there is some $u \in \mathcal{BC}_\delta([0,T))$ such that

$$u(t) = \left(\rho - \int_0^\infty x f(t,x)\, dx\right) \vee (\Phi_0 + \delta)$$

for all $t \in [0,T)$, where f is given by Eq. (2.11) in terms of u. Recall that $u(0) = u^{\mathrm{in}} > \Phi_0 + 2\delta$; thus, there exists t^* such that $u(t) \geq \Phi_0 + \delta$ for all $t \in [0,t^*]$ and hence for all $t \in [0,t^*]$,

$$u(t) = \rho - \int_0^\infty x f(t,x)\, dx\,.$$

Repeating this procedure we can construct an increasing sequence of times $\{t_n\}$ such that we have a solution f to our problem up to time t_n. We address now the maximality of this construction. Assume that the limit of $\{t_n\}$ is finite and let us denote it by T. We show now that $\lim_{t\to T^-} u(t) = \Phi_0$ by a contradiction argument. Let us assume that u does not converge to Φ_0 at T. We would have a solution f to the problem on $[0,T)$; however, since u' is bounded on $(0,T)$, u would have a limit at T^-. We would clearly have $\lim_{t\to T^-} u(t) > \Phi_0$. This allows us to extend f by continuity in T and $f(T,\cdot)$ would belong to $L^1((0,\infty),(1+x)dx)$. In that case we can apply the fixed point procedure once more to obtain a solution on $[T, T+t^*)$ for some $t^* > 0$, which contradicts the construction of the sequence $\{t_n\}$. Therefore, either $T = \infty$ or $\lim_{t\to T^-} u(t) = \Phi_0$. This ends the proof of Theorem 1.4. □

3.2. **Uniqueness.** In this section we let $T > 0$, $\rho > 0$, $\{a, b, \mathfrak{n}\}$ admissible kinetic rates and two non-negative functions f_1^{in} and f_2^{in} in $L^1((0,\infty),(1+x)\,dx)$. We consider two solutions f_1 and f_2 to Eq. (1.1)-(1.2)-(1.3) in the sense of Definition 1.2 on $(0,T)$ with mass ρ, rates $\{a, b, \mathfrak{n}\}$ and initial data f_1^{in} and f_2^{in} respectively. Let u_1 and u_2 given by the mass conservation (1.4) respectively with the solutions f_1 and f_2, and $v_1 = au_1 - b$ and $v_2 = au_2 - b$. We shall define the following, adapting the strategy from [21, theorem 2.3]:

$$F_i^+(t,x) = \int_x^\infty f_i(t,y)\,dy \tag{3.1}$$

for all $(t,x) \in \Omega_T$. In this section we use $\mathcal{D}(\Omega_T^*)$ the space of infinitely differentiable real-valued functions defined on Ω_T with compact support and $\mathcal{D}'(\Omega_T^*)$ its topological dual, the space of distributions on Ω_T^*.

Lemma 3.4. *We have, for $i = 1, 2$, that $F_i^+ \in L^\infty((0,T);L^1(0,\infty)) \cap L^\infty(\Omega_T^*)$, that $\partial_x F_i^+ = -f_i$ belongs to $L^\infty((0,T);L^1((0,\infty),(1+x)\,dx))$ and also that $\partial_t F_i^+$ belongs to $L^\infty((0,T);L^1(0,\infty))$. Moreover, they satisfy*

$$\int_0^\infty F_i^+(t,x)\,dx = \int_0^\infty x f_i(t,x)\,dx \tag{3.2}$$

for all $t \in (0,T)$, and

$$\partial_t F_i^+ + v_i \partial_x F_i^+ = 0\,, \quad \text{in } \mathcal{D}'(\Omega_T^*)\,. \tag{3.3}$$

Proof. Recall that f_i belongs to $L^\infty((0,T);L^1((0,\infty),(1+x)\,dx))$. The boundedness of F_i is an obvious consequence of the integrability of f_i and the definition in Eq. (3.1), as well as the regularity of the derivative in x. Integrability of F_i and formula (3.2) follows from Tonelli's Theorem. Eq. (3.3) is obtained using test functions of the form $\varphi(t,x) = \int_0^x \psi(t,y)\,dy$, for $\psi \in \mathcal{D}(\Omega_T^*)$, in Eq. (1.5) together with Fubini's theorem. Finally the regularity of the time derivatives follows from Eq. (3.3), the sublinearity of the rates and the regularity of $\partial_x F_i^+ = f_i$. □

From now on we denote

$$E^+ := F_1^+ - F_2^+ \text{ and } w = u_1 - u_2\,.$$

By Lemma 3.4 we deduce

$$\partial_t E^+ = -v_1 \partial_x F_1 + v_2 \partial_x F_2 = -v_1 \partial_x E^+ + a w f_2\,, \quad \text{in } \mathcal{D}'(\Omega_T^*)\,. \tag{3.4}$$

We are interested in estimates on the difference w; namely, for $t \in (0,T)$,

$$|w(t)| = \left| \int_0^\infty x f_1(t,x)\,dx - \int_0^\infty x f_2(t,x)\,dx \right|\,.$$

In virtue of (3.2), E^+ belongs to $L^\infty((0,T);L^1(0,\infty))$ and

$$|w(t)| \le \int_0^\infty |E^+(t,x)|\,dx \tag{3.5}$$

for all $t \in (0,T)$. We will also use that, thanks to Lemma 1.3,

$$F_i^+(t,0) = F_i^+(0,0) + \int_0^t \mathfrak{n}(u_i(s))\,dt$$

for $i = 1$, 2 and hence by (1.9)

$$|E^+(t,0)| \le |E^+(0,0)| + K_{\mathfrak{n}} \int_0^t |w(s)|\,ds \tag{3.6}$$

where $K_{\mathfrak{n}}$ is the Lipschitz constant of $\mathfrak{n}$ on $[\Phi_0,\rho]$. Our objective now is to perform a Gronwall estimate on E^+ by means of (3.4). By Lemma 3.4 and Eq. (3.4), for any real function β defined on $\mathbb{R}$, continuously differentiable with bounded derivatives, we have

$$\partial_t \beta(E^+) = -v_1 \partial_x \beta(E^+) + a w f_2 \beta'(E^+)\,, \quad \text{in } \mathcal{D}'(\Omega_T^*)\,.$$

In particular we are led to

$$\frac{d}{dt}\int_0^\infty \varphi(x)\beta(E^+(t,x))\,dx = \int_0^\infty \partial_x[v_1(t,x)\varphi(x)]\beta(E^+(t,x))\,dx + \int_0^\infty a(x)w(t)f_2(t,x)\beta'(E^+(t,x))\varphi(x)\,dx$$

for all φ belonging to $\mathcal{D}(0,\infty)$. Note that the distributional derivative $\partial_t\beta(E^+)$ belongs to $L^\infty(0,T)$. This is due to E^+ being bounded, f_2 being integrable against $(1+x)$, the sublinearity of a in (1.8), the fact that u_1 and u_2 are bounded and the boundedness of β'. We obtain

$$\int_0^\infty \varphi(x)\beta(E^+(t,x))\,dx \le \int_0^\infty \varphi(x)\beta(E^+(0,x))\,dx + \int_0^t\int_0^\infty \partial_x[v_1(s,x)\varphi(x)]\beta(E^+(s,x))\,dx\,dt + \|\beta'\|_{L^\infty}\int_0^t |w(s)| \int_0^\infty a(x)|\varphi(x)|f_2(s,x)\,dx\,dt \tag{3.7}$$

for φ belonging to $\mathcal{D}(0,\infty)$. The idea now is to replace β by the absolute value and to choose a suitable φ that helps us to deal with the difficulty that $\partial_x v_1$ is not bounded at zero. The latter is done in the next lemma.

Starting from the monotonicity assumption (H8), we distinguish two alternative (non mutually exclusive) possibilities:

$$\exists C>0, x^*>0 \text{ such that } \forall x \in (0,x^*), -\Phi'(x) < \frac{C}{a(x)}\,, \tag{H8a}$$

$$\exists C>0, x^*>0 \text{ such that } \forall x \in (0,x^*), -\Phi'(x) > \frac{C}{a(x)}\,, \tag{H8b}$$

It is clear that assumption (H8) implies that at least one of the two cases (H8a) or (H8b) holds true. Conversely, (H8a) and (H8b) together allow for a more general set of kinetic rates than (H8) alone does. We are going to show in the sequel that any of these two hypotheses guarantees uniqueness.

Lemma 3.5. *Let φ be defined as follows:*

(1) If assumption (H8a) *is true, we define*

$$\varphi(x) = \begin{cases} \frac{1}{a(x)}\,, & x \le \bar{x} \\ \frac{1}{a(\bar{x})}\,, & x > \bar{x} \end{cases} \tag{3.8}$$

for some given $\bar{x}$ (to be chosen later).

(2) If assumption (H8b) *holds true, we define*

$$\varphi(x) = \begin{cases} \frac{1}{a(x)} \exp\left(-\int_x^{\bar{x}} \frac{C/a(y)+\Phi'(y)}{\delta}\, dy\right)\,, & x \le \bar{x} \\ \frac{1}{a(\bar{x})}\,, \quad x > \bar{x} \end{cases} \tag{3.9}$$

for some given $\bar{x}$, C, δ (to be chosen later).

In both cases, with φ defined either in (3.8) *or* (3.9)*, we may choose the constant $\bar{x}$ (and C, δ in the second case) in a way that there exists a constant $K > 0$ such that, for all $x > 0$ and all $t \in (0, T)$,*

$$\partial_x[v_1(t,x)\varphi(x)] \le K\varphi(x). \tag{3.10}$$

Moreover, φ is continuous on $(0,\infty)$ and continuously differentiable for all $x > 0$ except at $\bar{x}$. It is bounded from below by

$$\varphi(x) \ge 1/\|a\|_{L^\infty(0,\bar{x})}\,, \tag{3.11}$$

and $a\varphi$ is bounded from above on $(0, \bar{x})$ by

$$a(x)\varphi(x) \le \max\left\{1, \exp\left(-\int_0^{\bar{x}} \frac{C/a(y)+\Phi'(y)}{\delta}\, dy\right)\right\}\,, \quad x < \bar{x}\,.$$

Proof. Note that finding a constant $K > 0$ such that Eq. (3.10) holds is equivalent to finding a constant $C > 0$ such that

$$(u_1(t) - \Phi(x))\partial_x(a\varphi)(x) \le (C + a\Phi')\varphi(x)\,.$$

Let us check that this inequality holds true for the function φ defined in (3.8) or (3.9) and well chosen constants.

We first deal with case 1. Let C and x^* be defined from assumption (H8a). For any $0 < \bar{x} \le x^*$, and for all $x \le \bar{x}$, the function φ defined in (3.8) satisfies

$$(u_1(t) - \Phi(x))\partial_x(a\varphi)(x) = 0 \le (C' + a(x)\Phi'(x))\,\varphi(x)\,,$$

for any $C' > C$, due to (H8a) and the fact that φ is positive. For $x > \bar{x}$,

$$\partial_x[v_1(t,x)\varphi(x)] \le \varphi(\bar{x})\left(\|a'(x)\|_{L^\infty(\bar{x},\infty)}\rho + \|b'(x)\|_{L^\infty(\bar{x},\infty)}\right) \le C''\varphi(x)\,,$$

for any constant $C'' \ge \left(\|a'(x)\|_{L^\infty(\bar{x},\infty)}\rho + \|b'(x)\|_{L^\infty(\bar{x},\infty)}\right)$. Thus Eq. (3.10) holds true for any x and for a sufficiently large constant K.

Now let us deal with case 2. Let C and x^* be defined from assumption (H8b). From lemma 2.5, let $\delta > 0$ and $x_0 > 0$ such that

$$\inf_{t\in(0,T)} u_1(t) > \sup_{x\in(0,x_0)} \Phi(x) + \delta\,.$$

Let then $\bar{x} = \min(x^*, x_0)$. For $x \le \bar{x}$, then φ satisfies

$$\begin{aligned}(u_1(t) - \Phi(x))\partial_x(a\varphi)(x) &= (u_1(t) - \Phi(x))\frac{C/a(x)+\Phi'(x)}{\delta}(a\varphi)(x) \\ &\le (C + a(x)\Phi'(x))\,\varphi(x) \le (C' + a(x)\Phi'(x))\,\varphi(x)\,,\end{aligned}$$

for any $C' \ge C$, as $u_1(t) - \Phi(x) \ge \delta$ but $C/a(x) + \Phi'(x) < 0$ and φ is positive. The case $x > \bar{x}$ is done similarly as in the case 1 above. $\square$

Note that as the function φ is bounded from below by (3.11), we have for some constant $C > 0$,

$$\int_0^\infty |E^+(t,x)|\,dx \le C\int_0^\infty \varphi(x)|E^+(t,x)|\,dx, \tag{3.12}$$

which will be used later on.

We need to introduce an auxiliary but classical result on moment propagation [6, 21] before closing the Gronwall loop.

Lemma 3.6. *Assume to be given the kinetic rates* $\{a, b, \mathfrak{n}\}$ *satisfying hypotheses* (H1)–(H2) *and* (H5)*, a constant* $\rho > \Phi_0$ *and a non-negative function* f^{in} *satisfying* (H9)*. Let* $T > 0$ *and* f *be a solution to the Lifshitz-Slyozov equation in the sense of Definition 1.2 on* $[0,T)$ *with mass* ρ*, kinetic rates* $\{a, b, \mathfrak{n}\}$ *and initial value* f^{in}*. Then, we have*

$$\sup_{t\in[0,T)}\int_0^\infty x^2 f(t,x)\,dx < \infty.$$

Proof. Set $h_R(x) = \min(x^2, R)$ for all $x > 0$ and $R > 0$ and plug h_R as test function in Eq. (1.6) thanks to Lemma 1.3. The results readily follows thanks to the mass conservation (1.4), bounds (1.8), Gronwall's lemma and finally taking $R \to \infty$. □

Lemma 3.7. *Let assumptions* (H8) *and* (H9) *hold true, and let* φ *defined in Lemma 3.5 above. Then, there exists* $C > 0$ *such that*

$$\int_0^\infty \varphi(x)|E^+(t,x)|\,dx \le \int_0^\infty \varphi(x)|E^+(0,x)|\,dx + C\int_0^t\int_0^\infty \varphi(x)|E^+(s,x)|\,dx\,dt + C\int_0^t |w(s)|\,dt + C\int_0^t |E^+(s,0)|\,dt\,.$$

Proof. For the most part this follows from a regularization procedure. To be able to use the function φ constructed in Lemma 3.5 into Eq. (3.7), we need to regularize it to make it C^∞ and to truncate its support. For each $R > 1$, denote by χ_R a real function in $\mathcal{D}(\mathbb{R})$ with $0 \le \chi_R \le 1$, such that $\chi_R = 1$ on $(1/R, R)$, with compact support in $[1/2R, R+1)$, $|\chi_R'| \le 4R$ on $(1/2R, 1/R)$, and $|\chi_R'| \le 2$ on $(R, R+1)$. Let $\{g^\varepsilon\}$ be a standard mollifying sequence. For the time being assume β is a non-negative function on $\mathbb{R}$, continuously differentiable with $|\beta'| \le 1$ and $\beta(0) = 0$. Define $\varphi_R^\varepsilon = \varphi_R * g^\varepsilon$ with $\varphi_R = \varphi\chi_R$ on $(0,\infty)$ and $\varphi_R(0) = 0$ otherwise. Consequently, φ_R^ε converges uniformly to φ_R on $\mathbb{R}$ as $\varepsilon \to 0$. Moreover, if we let $\varepsilon_0 < 1/2R$, then for all $\varepsilon \in (0, \varepsilon_0)$, the support of φ_R^ε is contained in $[1/2R-\varepsilon_0, R+1+\varepsilon_0]$. Observe that $\beta(y) \le |y|$ for all $x \in \mathbb{R}$; since $|E^+(t,x)|$ is bounded on Ω_T, it follows that

$$\lim_{\varepsilon\to 0}\int_0^\infty \varphi_R^\varepsilon(x)\beta(E^+(t,x))\,dx = \int_0^\infty \varphi_R(x)\beta(E^+(t,x))\,dx < \infty\,,$$

for any $t \in [0,T)$. Then, since f_2 belongs to $L^\infty\big((0,T); L^1((0,\infty);(1+x)\,dx)\big)$ and a is sublinear by (1.8), we have

$$\lim_{\varepsilon\to 0}\int_0^t\int_0^\infty a(x)f_2(t,x)\varphi_R^\varepsilon(x)\,dx\,dt = \int_0^t\int_0^\infty a(x)f_2(t,x)\varphi_R(x)\,dx\,dt$$

for all $t \in (0,T)$. Now, we remark that

$$\int_0^t \int_0^\infty \partial_x (v_1(s,x)\varphi_R^\varepsilon(x))\beta(E^+(s,x))\,dx\,dt$$
$$= \int_0^t \int_0^\infty \{\partial_x v_1(s,x)\varphi_R^\varepsilon(x) + v_1(s,x)\varphi_R' * g^\varepsilon(x)\}\,\beta(E^+(s,x))\,dx\,dt.$$

On one hand, as a and b are continuously differentiable on $(0,\infty)$, φ_R is compactly supported and E^+ belongs to $L^\infty(\Omega_T)$, we have

$$\lim_{\varepsilon\to 0}\int_0^t \int_0^\infty \partial_x v_1(s,x)\varphi_R^\varepsilon(x)\beta(E^+(s,x))\,dx\,dt$$
$$= \int_0^t \int_0^\infty \partial_x v_1(s,x)\varphi_R(x)\beta(E^+(s,x))\,dx\,dt.$$

On the other hand, note that $(\varphi\chi_R)'$ is piecewise continuous with compact support, and $(\varphi\chi_R)' * g^\varepsilon$ converges to $(\varphi\chi_R)'$ everywhere except at $\bar{x}$. But $\varphi\chi_R$ has compact support and a and b are continuous, hence bounded on this support. Moreover, E^+ belongs to $L^\infty(\Omega_T)$, so, via a dominated convergence theorem we have

$$\lim_{\varepsilon\to 0}\int_0^t \int_0^\infty v_1(s,x)\varphi_R'(x) * g^\varepsilon(x)\beta(E^+(s,x))\,dx\,dt$$
$$= \int_0^t \int_0^\infty v_1(s,x)\varphi_R'(x)\beta(E^+(s,x))\,dx\,dt.$$

Recapitulating, using that $\varphi_R \le \varphi$, that $f_2 \in L^\infty(L^1((1+x)dx))$ and that $a\varphi$ is bounded on $(0,\bar{x})$, we get

$$\int_0^\infty \varphi_R(x)\beta(E^+(t,x))\,dx \le \int_0^\infty \varphi(x)|E^+(0,x)|\,dx$$
$$+ \int_0^t \int_0^\infty \partial_x[v_1(s,x))\varphi_R(x)]\beta(E^+(s,x))\,dx\,dt$$
$$+ (\|a\varphi\|_{L^\infty(0,\bar{x})} + \varphi(\bar{x}))\|(1+x)f_2\|_{L^\infty(L^1)}\int_0^t |w(s)|\,dt. \tag{3.13}$$

Using Lemma 3.5 we deduce that there exists a constant $C>0$ such that

$$\partial_x[v_1(t,x))\varphi_R(x)] = \partial_x[v_1(t,x)\varphi(x)]\chi_R(x) + v_1(t,x)\varphi(x)\chi_R'(x)$$
$$\le C\varphi(x)\chi_R(x) + 4Rv_1(t,x)\varphi(x)\mathbf{1}_{(0,1/R)}(x) + +2v_1(t,x)\varphi(x)\mathbf{1}_{(R,\infty)}(x). \tag{3.14}$$

Set $R > \bar{x}$. Thus, from (3.14),

$$\int_0^\infty \partial_x[v_1(s,x))\varphi_R(x)]\beta(E^+(s,x))\,dx \le C\int_0^\infty \varphi(x)\beta(E^+(s,x))\,dx$$
$$+ 4\|a\varphi\|_{L^\infty(0,\bar{x})}\|u_1(t)-\Phi\|_{L^\infty((0,T)\times(0,\bar{x}))}R\int_0^{1/R}|E^+(s,x)|\,dx$$
$$+ 2\varphi(\bar{x})K_r(\rho+1)\int_R^\infty (1+x)|E^+(s,x)|\,dx,$$

where K_r comes from Eq. (1.8). Note that $n^2 \int_n^\infty f(t,x)\,dx \le \int_n^\infty x^2 f(t,x)\,dx$ thus

$$\int_R^\infty x F_i(t,x)\,dx = -\frac{1}{2}R^2 \int_R^\infty f_i(t,x)\,dx + \frac{1}{2}\int_R^\infty x^2 f_i(t,x)\,dx \to 0$$

as $R \to \infty$ for all t, from Lemma 3.6. Introducing the equation above into Eq. (3.13) and letting $R \to \infty$ we obtain, thanks to Lemma 3.6, that there is some constant C such that,

$$\int_0^\infty \varphi(x)\beta(E^+(t,x))\,dx \le \int_0^\infty \varphi(x)|E^+(0,x)|\,dx \\ + C\int_0^t \int_0^\infty \varphi(x)|E^+(s,x)|\,dx\,dt + C\int_0^t |E^+(s,0)|\,dt + C\int_0^t |w(s)|\,dt\,.$$

We then use the approximation of the absolute value $\beta(x) = |x| - \epsilon/2$ for $|x| > \varepsilon$ and $\beta(x) = \frac{1}{2\varepsilon}x^2$ for $|x| \le \varepsilon$ in the above equation and we let $\varepsilon \to 0$. This ends the proof.

□

Proof of theorem 1.6. We may now finish the proof of uniqueness. By Lemma 3.7, equations (3.5), (3.6) and (3.12) combined with Gronwall's lemma we have

$$|w(t)| + |E^+(t,0)| + \int_0^\infty \varphi(x)|E^+(t,x)|\,dx \\ \le C\left(|E^+(0,0)| + \int_0^\infty \varphi(x)|E^+(0,x)|\,dx\right)e^{CT}$$

and we conclude to Theorem 1.6 by taking $f_1^{\rm in} = f_2^{\rm in}$, so that $E^+(0,x) = 0$ for all $x \ge 0$ and then $u_1(t) = u_2(t)$, and $f_1(t,\cdot) = f_2(t,\cdot)$. □

3.3. **Criteria for global and local solutions.** In this section we prove Theorem 1.7, stating criteria for: (i) the existence of global solutions and (ii) the existence of local solutions for which u reaches the value Φ_0 in finite time. Recall that for a solution on $[0,T)$, by Lemma 1.3 we have that

$$\frac{du(t)}{dt} = \int_0^\infty a(x)(\Phi(x) - u(t))f(t,x)\ dx \tag{3.15}$$

is continuous on $[0,T)$. We exploit this formulation in the current section. For that aim let us introduce

$$\Phi^{sup} := \sup_{x\ge 0}\Phi(x), \quad \Phi_{inf} := \inf_{x\ge 0}\Phi(x).$$

Note that $0 \le \Phi_{inf} \le \Phi_0 \le \Phi^{sup}$, where Φ^{sup} need not be finite.

Lemma 3.8. *Let f be a solution of* (1.1)-(1.3) *in the sense of Definition 1.2 on* $[0,T)$. *Then the following assertions hold true during the life span of the solution:*

- *For $t \in [0,T)$, $u(t) \ge \Phi^{sup}$ implies that $\dot u(t) \le 0$,*
- *Assume that there is some $\bar t \ge 0$ such that $u(\bar t) \in [\Phi_{inf}, \Phi^{sup}]$. We have $u(t) \in [\Phi_{inf}, \Phi^{sup}]$ for every $t \ge \bar t$ in the life span of the solution.*
- *Assume that the solution is global (i.e. $T = \infty$). Then, provided that $\rho \ge \Phi_{inf}$, both $\liminf_{t\to\infty} u(t)$ and $\limsup_{t\to\infty} u(t)$ belong to $[\Phi_{inf}, \Phi^{sup}]$.*

Proof. All statements follow easily from Eq. (3.15). □

Proof of Theorem 1.7. First point. Since $\Phi(x) \geq \Phi_0$,

$$\frac{d(u(t)-\Phi_0)}{dt} = \frac{du(t)}{dt} \geq -(u(t)-\Phi_0)\int_0^\infty a(x)f(t,x)\,dx.$$

This entails

$$u(t)-\Phi_0 \geq (u(0)-\Phi_0)\exp\left(-\int_0^t\int_0^\infty a(x)f(t,x)\,dx\right).$$

Thanks to the control on the zeroth and first moments of f in Lemma 2.12 and the linear bound on the kinetic rate a in (1.8), we deduce that

$$u(t)-\Phi_0 \geq (u(0)-\Phi_0)\exp\left(-T\sup_{t\in(0,T)}\int_0^\infty a(x)f(t,x)\,dx\right) > 0\,.$$

This implies that $u(t) > \Phi_0$ as long as the solution makes sense. Hence the solution is global, because if it were to stop at time $T < \infty$, u would have a limit at T satisfying $u(T) > \Phi_0$ and therefore the solution could be extended further, which is a contradiction.

Second point. On one hand we have

$$\Phi(x) \leq \Phi_0 + \frac{\Phi(z)-\Phi_0}{z}x$$

for all $0 < x < z$. Also, suppose f^{in} has compact support in $[0, x_0]$. Note that

$$X(t;0,x_0) \leq x_0 + \overline{a}\rho t$$

and hence denoting by $z(t) = x_0 + \overline{a}t$, for all t, $f(t,\cdot)$ has support in $[0, z(t)]$. Hence we have

$$\begin{aligned}\frac{du(t)}{dt} &= \int_0^{z(t)} a(x)(\Phi(x)-u(t))f(t,x)\,dx\\ &\leq \int_0^{z(t)}\left[\Phi_0 - u(t) + \frac{\Phi(z(t))-\Phi_0}{z(t)}x\right]a(x)f(t,x)\,dx\,.\end{aligned}$$

Suppose that the solution is global, that is $u(t) > \Phi_0$ for all $t \geq 0$, and remarking that $\Phi(z(t)) \leq \Phi(x_0) < \Phi_0$, we have

$$\frac{du(t)}{dt} \leq -\frac{\Phi_0-\Phi(x_0)}{z(t)}\underline{a}\int_0^{z(t)} xf(t,x)\,dx\,.$$

Using mass conservation,

$$\frac{du(t)}{dt} \leq -\frac{\Phi_0-\Phi(x_0)}{z(t)}\underline{a}(\rho-u(t)) \leq 0\,.$$

Hence u decreases and $\rho - u(t) \geq \rho - u(0)$. Then we conclude

$$\frac{du(t)}{dt} \leq -K\frac{1}{z(t)} = -\frac{K}{x_0+\overline{a}t}$$

where $K = \underline{a}(\Phi_0 - \Phi(x_0))(\rho - u(0))$. Integrating this last estimate we obtain

$$u(t) \leq u(0) - \frac{K}{\overline{a}}\ln\left(1+\frac{\overline{a}}{x_0}t\right).$$

Therefore u reaches the value of Φ_0 in finite time, which yields a contradiction. □

4. Annex

As in Section 2, we assume here that we are given a function $u \in \mathcal{BC}^+_\rho([0,T))$, that $\rho > \overline{u} \geq \underline{u}_T > \Phi_0$ and that assumptions (H1)-(H5) hold.

4.1. **Proof of Lemma 2.7.** *Step 1. Proof that σ_t is non-increasing.* Let $t \in [0,T)$ and $0 < x < y$. If $\sigma_t(y) = 0$ then $\sigma_t(x) \geq \sigma_t(y) = 0$ by definition. Assume now that $\sigma_t(y) > 0$; we prove the monotonicity in this case by a contradiction argument. Therefore, let us assume that $\sigma_t(y) > \sigma_t(x)$. By Remark 2.4, $X(s;t,x) < X(s;t,y)$ for all $s \in (\sigma_t(x) \vee \sigma_t(y), T)$ and we have

$$0 \leq \lim_{s \to \sigma_t(x) \vee \sigma_t(y)} X(s;t,x) \leq \lim_{s \to \sigma_t(x) \vee \sigma_t(y)} X(s;t,y)\,.$$

Now thanks to our assumption $\sigma_t(y) > \sigma_t(x)$ and Lemma 2.2, we obtain

$$0 \leq X(\sigma_t(y);t,x) \leq \lim_{s \to \sigma_t(y)} X(s;t,y) = 0\,,$$

which entails $X(\sigma_t(y);t,x) = 0$. But this contradicts the definition of the maximal interval $J_{t,x}$. Thus, $\sigma_t(x) \geq \sigma_t(y)$ as desired.

Step 2. Proof that $x_c(t)$ is finite i.e. that $\{x > 0 \mid \sigma_t(x) = 0\}$ is not empty. Let $t \in [0,T)$, $y > 0$ and $x = X(t;0,y) > 0$. By the semigroup property, we have that $X(s;t;x) = X(s;0,y)$ for all $s \in (\sigma_t(x),T)$. Since $X(s;0,y)$ has maximal interval $[0,T)$, the trajectory $s \mapsto X(s;t,x)$ is defined on $[0,T)$ and hence $\sigma_t(x) = 0$. This proves that $\{x > 0 \mid \sigma_t(x) = 0\}$ is not empty, thus $x_c(t)$ is finite and non-negative.

Step 3. Separation by $x_c(t)$. Let $\{x^n\} \subset \{x > 0 \mid \sigma_t(x) = 0\}$ be a non-increasing sequence converging to $x_c(t)$. Since σ_t is non-increasing then $0 \leq \sigma_t(x) \leq \sigma_t(x^n) = 0$ for any $x > x^n$. Thus $(x_c(t),\infty) = \cup_{n\geq 1}(x^n,\infty) \subset \{x > 0 \mid \sigma_t(x) = 0\}$. However, if x is such that $\sigma_t(x) = 0$ then $x \geq x_c(t)$ by definition, which proves that the former inclusion is an equality. Moreover, if $x_c(t) > 0$, for any $x \in (0, x_c(t))$ we have $\sigma_t(x) > 0$ by the construction of $x_c(t)$.

Step 4. $x_c(t)$ is positive. Let $t \in (0,T)$. It is sufficient to construct some $x > 0$ satisfying $\sigma_t(x) > 0$, as then $x_c(t) \geq x > 0$. Let $\delta > 0$ and $x_0 > 0$ given by Lemma 2.5. Thus, for all $x \in (0,x_0)$ and $s \in (\sigma_t(x), t)$, since $X(t;t,x) = x < x_0$ we have,

$$B(t;t,A(x)) - B(s;t,A(x)) \geq \delta(t-s), \quad \text{that is,} \quad B(s;t,A(x)) \leq A(x) - \delta(t-s)\,.$$

Now we are ready to conclude by a contradiction argument. Contrary to what we want, suppose that $\sigma_t(x) = 0$ for all $x \in (0,x_0)$. This enables us to compute the limit $\lim_{s\to 0} B(s;t,A(x)) \leq A(x) - \delta t$ for all $x \in (0,x_0)$. This entails that there exists $x_1 \in (0,x_0)$ such that

$$\lim_{s \to 0} B(s;t,A(x_1)) \leq A(x_1) - \delta t < 0\,,$$

since A is continuous and $A(0) = 0$. This contradict that, for all $s \in (0,t)$, $B(s;t,A(x_1)) = A(X(s;t,x_1)) > 0$, after Lemma 2.2. Thus, there is some $x > 0$ such that $\sigma_t(x) > 0$. This concludes the proof of Lemma 2.7.

4.2. **Proof of Proposition 2.8.** We will prove a slightly more general result.

Lemma 4.1. *For each $t \in (0,T)$ and $s \in [0,t)$ the following statements hold true:*
(1) We have $X(t;s,0^+) := \lim_{x\to 0^+} X(t;s,x) \in (0,x_c(t))$.
(2) The map $x \mapsto X(t;s,x)$ is an increasing $\mathcal{C}^1$-diffeomorphism from $(0,\infty)$ to $(X(t;s,0^+),\infty)$.

(3) The semigroup property $X(t;\tau,X(\tau;s,0^+)) = X(t;s,0^+)$ *holds for all* $\tau \in [s,T)$.
(4) We have $X(t;0,0^+) = x_c(t)$ *and* $X(t;s,x_c(s)) = x_c(t)$.
(5) The bound $x_c(t) \leq C(T)$ *holds for some positive constant* $C(T)$ *independent of* $t \in (0,T)$.

Remark. Proposition 2.8 is a straightforward aplication of this Lemma with $s = 0$.

Proof. Recall that $\underline{u}_T > \Phi_0$. Let $t \in (0,T)$ and $s \in [0,t)$. Lemma 2.7 ensures that $J_{0,x} = [0,T)$ and $J_{s,x} = (\sigma_s(x),T)$ for all $x > 0$. Then $x \mapsto X(t;s,x)$ is continuously differentiable on $(0,\infty)$ by the Cauchy–Lipschitz theory. Its derivative, given by (2.2), is strictly positive. Thus, the map $x \mapsto X(t;s,x)$ is strictly increasing and then a diffeomorphism onto its image. We prove below that $\lim_{x\to\infty} X(t;s,x) = +\infty$ for any $s \in [0,t)$, that $X(t;0,0^+) = x_c(t)$, that $0 < X(t,s,0^+) < x_c(t)$ for $s \in (0,t)$ and the semigroup properties. The bound on $x_c(t)$ in item *(5)* is a direct consequence of the bound (2.3) at the limit $x \to 0$.

Step 1. Proof of $\lim_{x\to\infty} X(t;s,x) = \infty$. We fix $y > 0$. Using the bound (2.3) with $x = X(t;s,y)$ we deduce that

$$X(\tau;t,X(t;s,y)) \leq C(T)(1 + X(t;s,y))\,, \text{ for all } \tau \in J_{s,y} = (\sigma_s(y),T) = (\sigma_t(x),T).$$

Setting $\tau = s$ we obtain

$$y \leq C(T)(1 + X(t;s,y))\,.$$

This concludes the proof by taking the limit $y \to \infty$.

Step 2. Proof that $X(t;s,0^+) > 0$. Let $\delta > 0$ and $x_0 > 0$ given in Lemma 2.5. Let x_n a positive decreasing sequence converging to zero. By Lemma 2.5, either $X(t;s,x_n) > x_0$ or $B(t;s,A(x_n)) - A(x_n) \geq \delta(t-s)$. Thus,

$$\lim_{n\to\infty} X(t;s,x_n) \geq \min(x_0, A^{-1}(\delta(t-s))) > 0\,.$$

Step 3. Proof of $X(t;0,0^+) = x_c(t)$. Let $t \in (0,T)$. We take a positive, non-increasing sequence $\{x^n\}$ converging to zero. As $x \mapsto X(t;0,x)$ is monotonically increasing we can define

$$\bar{x} := \lim_{x\to 0} X(t;0,x) = \lim_{x^n \searrow 0} X(t;0,x^n)\,.$$

Note that $\sigma_t(X(t;0,x^n)) = 0$ and then $X(t;0,x^n) \geq \bar{x} \geq x_c(t)$, as $x_c(t) = \inf\{x > 0 \mid \sigma_t(x) = 0\}$.

We prove that $\bar{x} = x_c(t)$ by a contradiction argument. Assume that

$$\bar{x} > x_c(t). \tag{4.1}$$

Let $y \in (x_c(t),\bar{x})$; we have $\sigma_t(y) = 0$ as $y > x_c(t)$. Since $y < \bar{x} \leq X(t;0,x^n)$ for any n, we obtain

$$\lim_{s\to 0} X(s;t,y) \leq \lim_{s\to 0} X(s;t,X(t;0,x^n)) = \lim_{s\to 0} X(s;0,x^n) = x^n\,.$$

Passing to the limit $n \to \infty$ we deduce that

$$\lim_{s\to 0} X(s;t,y) = 0\,, \text{ and hence } \lim_{s\to 0} B(s;t,A(y)) = 0$$

by the continuity of A at zero.

Consider now y_1, y_2 such that $x_c(t) < y_1 < y_2 < \bar{x}$. There holds that

$$B(s;t,A(y_2)) - B(s;t,A(y_1)) = \int_{y_1}^{y_2} \frac{d}{dz}\left(B(s;t,A(z))\right) dz = \int_{y_1}^{y_2} \left(\frac{d}{dy}B\right)(s;t,A(z))\,\frac{dz}{a(z)}\,. \quad (4.2)$$

We look for a lower bound on this quantity. Note that $\sigma_t(y_2) = 0$ and that $\lim_{s\to 0} X(s;t,y_2) = 0$. Thus, there exists s_0 such that $X(s_0;t,y) < X(s_0;t,y_2) < x_0$ for any $y \in (y_1, y_2)$. Now we use Lemma 2.6: for any $s \in (0, s_0)$ we have that

$$\int_s^{s_0} |(a\cdot\Phi')(X(r;t,y))|\,dr \le \frac{1}{\delta}\int_0^{x_1} |\Phi'(z)|\,dz,$$

and from Eq. (2.5) we deduce that for any $y \in (y_1, y_2)$,

$$\left(\frac{d}{dy}B\right)(s;t,A(y)) \ge \exp\left(-\frac{1}{\delta}\int_0^{x_1}|\Phi'(z)|\,dz - \int_{s_0}^{t}|(a\cdot\Phi')(X(r;t,y))|\,dr\right) := c_1 > 0\,. \quad (4.3)$$

Hence, as the lower bound on Eq. (4.3) is independent of s, letting $s \to 0$ in (4.2) we obtain

$$0 > c_1 \int_{y_1}^{y_2} \frac{1}{a(z)}\,dz\,,$$

which is a contradiction. Thus assumption (4.1) is absurd and therefore $X(t;0,0^+) = \bar{x} = x_c(t)$.

Step 4. Semi-group property in (4). Recall that $X(t;s,X(s;0,x)) = X(t;0,x)$ for all $x > 0$ and $s \in (0,t)$. By continuity we can pass to the limit $x \to 0$ so that $X(t;s,X(s;0,0^+)) = X(t;0,0^+)$. Thanks to *Step 3* this reads $X(t;s,x_c(s)) = x_c(t)$.

Step 5. Proof that $X(t;s,0^+) < x_c(t)$ and the semi-group property in point (3). Let $t \in (0,T)$, $s \in (0,t)$ and $x < x_0$. To start we point out that the limit $X(t,s,0^+)$ exists since $x \mapsto X(t;s,x)$ is positive and monotonically increasing. By Lemma 2.5, there holds that $X(s;0,x) > \min(x, x_0) = x$ for any $s \in (0,t)$. Thus, by Remark 2.4,

$$X(t;0,x) = X(t;s,X(s;0,x)) \ge X(t;s,x)\,.$$

Taking $x \to 0^+$, we deduce that $x_c(t) = X(t;0,0^+) \ge X(t;s,0^+)$. Now we recall that $X(s;0,0^+) = x_c(s) > 0$ by Lemma 2.7. Thus, for $y \in (0, x_c(s))$ we have, by Remark 2.4,

$$X(t;s,0^+) \le X(t;s,y) \le X(t;s,x_c(s)) = x_c(t).$$

Then it follows that $x_c(t) > X(t;s,0^+)$ as desired: if we had $x_c(t) = X(t;s,0^+)$ for some $s > 0$, we would deduce that $X(t;s,y) = x_c(t)$ for any $y \in (0, x_c(s))$, which contradicts that $x \mapsto X(t;s,x)$ is a diffeomorphism onto $(0,\infty)$. Finally, since $X(t;\tau,X(\tau;s,x)) = X(t;s,x)$ for all $\tau \in (s,T)$, by continuity at the limit $x \to 0$ we obtain the semi-group property in point (3). □

A useful consequence of Lemma 4.1 is the next lemma,

Lemma 4.2. *For any $0 < s_1 < s_2 < t$ we have $X(t;s_2,0^+) < X(t;s_1,0^+)$.*

Proof. This can be shown by a contradiction argument; let us first assume that $X(t;s_2,0^+) > X(t;s_1,0^+)$. Since $x \mapsto X(t;s_1,x)$ is a diffeomorphism from $(0,\infty)$ to $(X(t;s_1,0^+),\infty)$, there exists $x>0$ such that $X(t;s_1,x) = X(t;s_2,0^+)$. Now we have that $X(s_2;t,X(t;s_2,0^+)) = 0$ thanks to semigroup properties, but we also have

$$X(s_2;t,X(t;s_2,0^+)) = X(s_2;t,X(t;s_1,x)) = X(s_2;s_1,x) > 0$$

by Lemma 2.5 and we reach a contradiction. In the case $X(t;s_2,0^+) = X(t;s_1,0^+)$ we conclude that $0 = X(s_2;t,X(t;s_2,0^+)) = X(s_2;s_1,0^+) > 0$ by Lemma 4.1, a contradiction again. □

4.3. **Proof of Proposition 2.9.** The proof of Proposition 2.9 being somewhat technical, we separate it in some lemmas so that the Proposition is obtained as a straightforward recollection of the results stated in these lemmas.

Lemma 4.3. *For all $t\in(0,T)$, σ_t is (strictly) decreasing on $(0,x_c(t))$ and*

$$\lim_{y\to 0^+} X(t,\sigma_t(x),y) = x \quad \text{for any} \quad x\in(0,x_c(t)). \tag{4.4}$$

Proof. Let $\delta>0$ and x_0 given by Lemma 2.5. Fix $t\in(0,T)$ and $0<y<x<x_c(t)$. We know from Lemma 2.7 that $\sigma_t(y)\geq\sigma_t(x)$ and we will prove that equality cannot hold. Let us assume that $\sigma_t(x)=\sigma_t(y)$ and argue by contradiction. By Lemma 2.2, there exists $s_0\in(\sigma_t(x),t)$ such that $X(s_0;t,x)<x_0$ and by Remark 2.4 we also have $X(s_0;t,z)<x_0$ for all $z\in(y,x)$. By Lemma 2.6,

$$\left|\int_s^t (a\cdot\Phi')(X(r;t,z))\,dr\right| \leq \frac{1}{\delta}\int_0^{x_0}|\Phi'(\xi)|\,d\xi + \int_{s_0}^t (a\cdot\Phi')(X(r;t,z))\,dr \tag{4.5}$$

for all $s\in(\sigma_t(x),s_0)$ and $z\in(y,x)$. Using Lemma 2.5 and the bound (2.3) we can now estimate

$$0 < x_m := X(s_0;t,y) \leq X(\tau;t,y) \leq X(\tau;t,z) \leq x_M := C(T)(1+x)$$

for all $z\in(y,x)$ and $\tau\in(s_0,t)$. We then have from Eq. (4.5),

$$\left|\int_s^t (a\cdot\Phi')(X(r;t,z))\,dr\right| \leq \frac{1}{\delta}\int_0^{x_0}|\Phi'(\xi)|\,d\xi + T\|a\cdot\Phi'\|_{L^\infty(x_m,x_M)}. \tag{4.6}$$

Finally, by Eqs (2.5) and (4.6) we deduce that

$$B(s;t,A(x)) - B(s;t,A(y)) \geq e^{\left(-\frac{1}{\delta}\int_0^{x_0}|\Phi'(\xi)|\,d\xi - T\|a\cdot\Phi'\|_{L^\infty(x_m,x_M)}\right)}\int_y^x \frac{1}{a(z)}\,dz\,.$$

We may now take the limit $s\to\sigma_t(x)=\sigma_t(y)$ thanks to Lemma 2.2 to deduce that $\int_y^x \frac{1}{a(z)}\,dz\leq 0$. This contradicts the strict positivity of a. Thus $\sigma_t(y)>\sigma_t(x)$ as desired.

We proceed now to the proof of the limit (4.4). Let $x\in(0,x_c(t))$ be such that $\sigma_t(x)>0$. We remark that by Lemma 4.1, the limit $X(t;\sigma_t(x),0^+)$ exists. As a first step we prove that this limit is greater or equal than x. Let x^n a positive decreasing sequence towards zero. Using Lemma 2.5, there exists $x_0>0$ such that $X(s;\sigma_t(x),x^n)\geq\min(x_n,x_0)>0$, for all $s\in(\sigma_t(x),T)$. Thus, $(\sigma_t(x),T)\subseteq J_{t,X(t;\sigma_t(x),x^n)}$. By definition of σ_t, we have

$$\sigma_t\left(X(t;\sigma_t(x),x^n)\right)\leq\sigma_t(x) \text{ and then } X(t;\sigma_t(x),x^n)\geq x$$

(otherwise it contradicts the fact that σ_t is decreasing). Moreover, by Remark 2.4, $X(t;\sigma_t(x),x^n)$ is a decreasing sequence. Hence it converges to some $y\geq x$.

We show now that $y = x$ arguing by contradiction. Assume that $y > x$. For $z \in (x, x_c(t) \wedge y)$ we have that $0 < \sigma_t(z) < \sigma_t(x)$ by the first part of the proof. Then Remark 2.4 yields, for all $s > \sigma_t(x)$ and $n \in \mathbb{N}$,

$$X(s;t,x) < X(s;t,z) < X(s;t,y) \leq X(s;t,X(t;\sigma_t(x),x^n)) = X(s;\sigma_t(x),x^n)\,.$$

Taking the limit $s \to \sigma_t(x)$ we obtain

$$0 \leq X(\sigma_t(x);t,z) \leq x^n\,.$$

Taking now the limit $n \to \infty$ leads to $X(\sigma_t(x);t,z) = 0$, which contradicts that $0 < \sigma_t(z) < \sigma_t(x)$ by Lemma 2.2. Then $y = x$, which concludes the proof. □

Lemma 4.4. *For all $t \in (0,T)$, σ_t is a (strictly) decreasing homeomorphism from $(0, x_c(t))$ to $(0,t)$.*

Proof. Step 1. Proof of $\sigma_t(x_c(t)) = 0$. By definition, $\sigma_t(x_c(t)) = \inf J_{t,x_c(t)}$, and $J_{t,x_c(t)} = \{s \geq 0; X(s;t,x_c(t)) > 0\}$. By point (*4*) in Lemma 4.1, we have $X(s;t,x_c(t)) = x_c(s)$. Thus, $J_{t,x_c(t)} = \{s \geq 0; x_c(s) > 0\}$. Thanks to Lemma 2.7, $x_c(s)$ is positive for any $s > 0$ and hence $\sigma_t(x_c(t)) = \inf J_{t,x_c(t)} = 0$.

In the next two steps, we characterize the sequential continuity of σ_t both at the left and the right. There is no loss of generality in restricting ourselves to monotone sequences.

Step 2. The map $x \mapsto \sigma_t(x)$ is continuous at the left. Let $x \in (0, x_c(t)]$ and $x^n \in (0, x_c(t))$ an increasing sequence converging to x with $x^n < x$ for all $n \geq 1$. By Lemma 4.3, $\sigma_t(x^n) > \sigma_t(x)$ for all $n \geq 1$ and $\{\sigma_t(x^n)\}$ is a decreasing sequence, thus it converges. To show the sequential continuity of σ_t we argue by contradiction; therefore we assume that $\sigma_t(x^n)$ converges to $s > \sigma_t(x)$. Note that in particular $\sigma_t(x^n) \geq s$. Lemma 2.5 provides us with $\delta > 0$ and x_0 such that $u(t) \geq \Phi_0 + \delta$ and $|\Phi(y) - \Phi_0| < \delta/2$ for all $y \in (0, x_0)$, so that $V(t, A(y)) = u(t) - \Phi(y) \geq \delta/2$ and then

$$\frac{dB(t;s,A(y))}{ds} \leq -\delta I(t;s,A(y))/2 \leq 0\,, \quad \text{for } s \in (\sigma_t(x), \sigma_t(x^n)).$$

Thus, as A^{-1} is increasing, we have that

$$X(t;\sigma_t(x^n),y) \leq X(t;s,y) \leq X(t;\sigma_t(x),y) \quad \text{for all } y \in (0,x_0).$$

Using Lemma 4.3 we may take the limit $y \to 0$ to obtain

$$x^n \leq X(t;s,0^+) \leq x.$$

Taking next the limit $n \to \infty$ we deduce that $X(t;s,0^+) = x = X(t;\sigma_t(x),0^+)$, which contradicts Lemma 4.2 since we had assumed that $s > \sigma_t(x)$. Therefore $\sigma_t(x^n)$ converges to $\sigma_t(x)$ as desired.

Step 3. The map $x \mapsto \sigma_t(x)$ is continuous at the right. Let $x \in (0, x_c(t))$ and take x^n a decreasing sequence converging to x and such that $x < x^n < x_c(t)$. Thus $\sigma_t(x^n)$ is increasing and $\sigma_t(x^n) < \sigma_t(x)$. We show again the sequential continuity by means of a contradiction argument. Assume that $\sigma_t(x^n) \to s < \sigma_t(x)$ and hence $\sigma_t(x^n) < s$. Similarly to Step 2, for any $y < x_0$ we have

$$X(t;\sigma_t(x^n),y) \geq X(t;s,y) > X(t;\sigma_t(x),y)\,.$$

Using Lemma 4.3 and taking the limit $y \to 0$ we obtain

$$x^n \geq X(t;s,0^+) \geq x\,.$$

Now we deduce that $X(t;s,0^+) = x$ taking the limit $n \to \infty$. But for $u \in (s, \sigma_t(x))$ we have $X(u;s,0^+) > 0$ and also

$$X(u;s,0^+) = X(u;t,X(t;s,0^+)) = X(u;t,x) > 0$$

by Lemma 4.1, which contradicts that $u < \sigma_t(x)$. This shows that $\sigma_t(x^n) \to \sigma_t(x)$ as $n \to \infty$.

Step 4. Proof of $\lim_{x\to 0} \sigma_t(x) = t$. Let x^n a positive decreasing sequence converging to zero. Then $\sigma_t(x^n) \le t$ is increasing and converges to $\bar{s} \in (0,t]$. We use a contradiction argument to prove that $\bar{s} = t$. Assume that $\bar{s} < t$. There exists $N \ge 1$ such that $x^n < \min(x_0, x_c(t))$ for all $n \ge N$. By Lemma 2.5,

$$A(x^n) - B(s;t,A(x^n)) \ge \delta(t-s) \quad \text{for all}\, s \in (\sigma_t(x^n), t)\, \text{and}\, n \ge N.$$

Then, for $\eta \in (0, t-\bar{s})$ we have that $B(t-\eta;t,A(x^n))$ is well-defined for all $n \ge N$ and

$$B(t-\eta;t,A(x^n)) \le A(x^n) - \delta\eta\,.$$

We can choose n large enough such that $A(x^n) - \eta\delta < 0$, since $A(x_n)$ converges to zero. This contradicts that $B(t-\eta;t,A(x^n)) > 0$ by construction. Thus we must have $\bar{s} = t$.

Step 5. Conclusion. Putting together what we have proved so far, σ_t is a strictly decreasing, positive and continuous function on $(0, x_c(t))$. Therefore it is an homeomorphism onto its image $(0,t)$ since $\lim_{x\to 0} \sigma_t(x) = t$, and, by continuity, $\lim_{x\to x_c(t)} \sigma_t(x) = 0$.

□

Lemma 4.5. *There holds that*

$$\sigma_t^{-1}(s) = \lim_{x\to 0} X(t;s,x) \quad \textit{for all}\; t \in (0,T) \quad \textit{and}\; s \in (0,t).$$

Moreover, there exists a constant $C(T)$*, independent on* $u \in \mathcal{BC}_\rho^+$*, such that*

$$\sigma_t^{-1}(s) \le C(T), \quad \textit{for all}\; t \in (0,T) \quad \textit{and}\; s \in (0,t). \tag{4.7}$$

Proof. Let $t \in (0,T)$ and $s \in (0,t)$. Consider a positive decreasing sequence $\{x^n\}$ converging to zero; note that $\{X(t;s,x^n)\}$ is a decreasing sequence. Thus, by Lemma 4.4, $\sigma_t(X(t;s,x^n))$ is increasing and verifies that $\sigma_t(X(t;s,x^n)) \le s$; hence $\sigma_t^{-1}(s) \le X(t;s,x^n)$. Since the sequence $\{X(t;s,x^n)\}$ decreases, it converges to some $\bar{x} \ge \sigma_t^{-1}(s)$ -and in particular $X(t;s,x^n) \ge \bar{x}$ for all $n \ge 1$. We argue by contradiction to show that equality holds. Assume $\bar{x} > \sigma_t^{-1}(s)$ and let $y \in (\sigma_t^{-1}(s), \bar{x})$, hence $\sigma_t(y) < s$ and $X(s;t,y) > 0$. But $y < \bar{x} \le X(t;s,x^n)$ and it follows that $X(s;t,y) < x^n \to 0$, which contradicts that $X(s;t,y) > 0$. Thus $\bar{x} = \sigma_t^{-1}(s)$ as claimed. The bound (4.7) is obtain by taking the limit $x \to 0$ in (2.3). □

Lemma 4.6. *For all* $t \in (0,T)$*,* σ_t^{-1} *is a decreasing* C^1*-diffeomorphism from* $(0,t)$ *to* $(0, x_c(t))$ *and its derivative is given by Eq.* (2.10)*.*

Proof. For any $t \in (0,T)$, $s \in (0,t)$ and $x > 0$ we have from Eq. (2.5) that

$$\begin{aligned} B(t;t,A(x)) - B(t;s,A(x)) &= \int_s^t \frac{dB(t;\tau,A(x))}{d\tau}\, d\tau. \\ &= -\int_s^t (u(\tau) - \Phi(x)) \exp\left(-\int_\tau^t (a\cdot\Phi')(X(r;\tau,x))\, dr\right) d\tau\,. \end{aligned} \tag{4.8}$$

We aim to take the limit $x \to 0^+$ in this equation, since as $x \to 0^+$ we have $B(t,t,A(x)) = A(x) \to 0$ and also $B(t;s,A(x)) = A(X(t;s,x)) \to A(\sigma_t^{-1}(s))$ from Lemma 4.5, as $x \to 0^+$. Interchanging the limit and the integral we get a formula for $\sigma_t^{-1}(s)$ that will enable us to compute the derivatives. We split this argument into three steps.

Step 1. We justify that for all $t \in (0,T)$ and $\tau \in (0,t)$,

$$\begin{aligned}\lim_{x\to 0^+} & \frac{dB(t;\tau,A(x))}{d\tau} \\ &= -\lim_{x\to 0^+} (u(\tau)-\Phi(x)) \exp\left(-\int_\tau^t (a\cdot\Phi')(X(r;\tau,x))\,dr\right) \\ &= -(u(\tau)-\Phi_0)\exp\left(-\int_\tau^t (a\cdot\Phi')(\sigma_r^{-1}(\tau))\,dr\right). \qquad (4.9)\end{aligned}$$

Let $t \in (0,T)$, $\tau \in (0,t)$. To prove the limit in (4.9), we will split the integral above in two parts,

$$\int_\tau^t (a\cdot\Phi')(X(r;\tau,x))\,dr = \int_\tau^{\tau_0} (a\cdot\Phi')(X(r;\tau,x))\,dr + \int_{\tau_0}^t (a\cdot\Phi')(X(r;\tau,x))\,dr\,. \qquad (4.10)$$

Here τ_0 is chosen small enough and independent of x, so that it allows to make the change of variable $r \mapsto z = X(r;\tau,x)$ (as in the proof of Lemma 2.6) on (τ,τ_0) and to be away from zero on (τ_0,t).

Let $\varepsilon > 0$ and consider $\delta > 0$ and $x_0 > 0$ given by Lemma 2.5. We recall that Φ' is integrable around zero thanks to (H4); therefore we can find $x_1 \in (0,x_0)$ such that $\int_0^{x_1} |\Phi'(z)|\,dz < \delta\varepsilon/2$. Let now $x_2 \in (0,x_1)$. There exists $\tau_0 \in (\tau,t)$ such that $X(\tau_0;\tau,x_2) < x_1$ by continuity in the first variable. Hence $X(\tau_0;\tau,x) < X(\tau_0;\tau,x_2) < x_1$ for all $x \in (0,x_2)$ by Remark 2.4. Using Lemma 2.6 with (2.9) we have that

$$0 \le \int_\tau^{\tau_0} a(X(r;\tau,x))|\Phi'(X(r;\tau,x))|\,dr \le \frac{1}{\delta}\int_x^{X(\tau_0;\tau,x)} |\Phi'(z)|\,dz \le \frac{\varepsilon}{2} \qquad (4.11)$$

for all $x \in (0,x_2)$. Also, by Fatou's lemma, as $x \to 0$,

$$\int_\tau^{\tau_0} a(\sigma_r^{-1}(\tau))|\Phi'(\sigma_r^{-1}(\tau))|\,dr \le \frac{\varepsilon}{2}. \qquad (4.12)$$

For the second term in Eq. (4.10), we bound the integrand uniformly in x. Indeed, by bound (2.3) there exists a constant $C(T) > 0$ independent of τ, such that $X(r;\tau,x) \le x_M := C(T)(1+x_0)$ for all $r \in (\tau,T)$ and $x \in (0,x_2)$. We also have

$$B(\tau_0;\tau,A(x)) \ge \delta(\tau_0-\tau) + A(x) \ge \delta(\tau_0-\tau) \qquad (4.13)$$

by Lemma 2.5. Thus, there holds that

$$x_0 > X(\tau_0;\tau,x) \ge x_m := A^{-1}(\delta(\tau_0-\tau)) > 0 \quad \text{for all}\, x \in (0,x_2),$$

which by Lemma 2.5 entails that

$$X(r;\tau,x) = X(r;\tau_0,X(\tau_0;\tau,x)) \ge X(\tau_0;\tau,x) \ge x_m$$

for all $r \geq \tau_0$ and for every $x \in (0, x_2)$. Hence, the map $r \mapsto (a \cdot \Phi')(X(r;\tau,x))$ is uniformly bounded in $r \in (\tau_0, t)$ and $x \in (0, x_2)$, which justifies that

$$\lim_{x\to 0^+} \int_{\tau_0}^{t} (a \cdot \Phi')(X(r;\tau,x))\, dr = \int_{\tau_0}^{t} (a \cdot \Phi')(\sigma_r^{-1}(\tau))\, d\tau. \tag{4.14}$$

Finally, by Eqs. (4.14), (4.11) and (4.12) we deduce that for all $\varepsilon > 0$,

$$\lim_{x\to 0^+} \left| \int_{\tau}^{t} (a \cdot \Phi')(X(r;\tau,x)) - (a \cdot \Phi')(\sigma_r^{-1}(\tau))\, dr \right| \leq \varepsilon$$

and in this way we conclude (4.9).

Step 2. We aim now to bound the derivative of B to pass to the limit $x \to 0$ in Eq. (4.8). Let $t \in (0, T)$. We now split the integral in (4.8) in three parts, again to separate the contributions where $X(r;\tau,x)$ is close to zero and away from it:

$$\begin{aligned}
\int_{s}^{t} (u(\tau) - \Phi(x)) \exp\left(- \int_{\tau}^{t} (a \cdot \Phi')(X(r;\tau,x))\, dr \right) d\tau& \\
= \int_{t_0}^{t} (u(\tau) - \Phi(x)) \exp\left(- \int_{\tau}^{t} (a \cdot \Phi')(X(r;\tau,x))\, dr \right) d\tau& \\
+ \int_{s}^{t_0} (u(\tau) - \Phi(x)) \exp\Big(- \int_{\tau}^{\tau+\tau_0} (a \cdot \Phi')(X(r;\tau,x))\, dr& \\
- \int_{\tau+\tau_0}^{t} (a \cdot \phi')(X(r;\tau,x))\, dr \Big)\, d\tau\,,&
\end{aligned}$$

for some $0 < t_0 < t_0 + \tau_0 < t$, with t_0 sufficiently close to t and τ_0 small enough, both independent from x, as we explain in what follows. Let $\delta > 0$ and $x_0 > 0$ given by Lemma 2.5 and let $x_1 \in (0, x_0)$. As $X(t;t,x_1) = x_1$, by continuity, there exists $t_0 < t$ such that $X(t;\tau,x_1) < x_0$ for all $\tau \in (t_0, t)$ and hence

$$X(r;\tau,x) < X(t;\tau,x) < X(t;\tau,x_1) < x_0\,, \text{ for all } r \in (\tau,t),\ x \in (0,x_1),\ \tau \in (t_0,t)\,.$$

Using Lemma 2.6, we get that

$$\left| \int_{\tau}^{t} |(a \cdot \Phi')(X(r;\tau,x))|\, dr \right| \leq \frac{1}{\delta} \int_{0}^{x_0} |\Phi'(z)|\, dz$$

for all $\tau \in (t_0, t)$ and $x \in (0, x_1)$.

Let now $\tau \in (0, t_0)$. Since $x_1 < x_0$ there exists $0 < \tau_0 < t - t_0$ such that $A(x_1) + \rho\tau_0 < A(x_0)$. Hence by Eq. (2.4),

$$B(\tau + \tau_0; \tau, A(x)) \leq A(x) + \rho\tau_0 \leq A(x_0) \quad \text{for all}\, x \in (0, x_1)$$

and by Lemma 2.5,

$$X(r;\tau,x) < X(\tau+\tau_0;\tau,x) < x_0 \quad \text{for all}\, r \in (\tau, \tau+\tau_0).$$

Note that the condition on τ_0 ensures $\tau + \tau_0 < t$ whenever $\tau \in (0, t_0)$. On the other hand we have

$$B(\tau + \tau_0; \tau, A(x)) \geq A(x) + \delta\tau_0 \geq \delta\tau_0$$

and, by Lemma 2.5, $X(r;\tau,x) \geq x_m := A^{-1}(\delta\tau_0)$ for all $r \in (\tau + \tau_0; t)$ and $x \in (0, x_1)$. Since $X(r;\tau,x) \leq x_M := C(T)(1 + x_1)$ for all $\tau \in (0, T)$, $r \in (\tau, T)$ and $x \in (0, x_1)$, we have

$$\left| \int_{\tau+\tau_0}^{t} (a \cdot \Phi')(X(r;\tau,x))\, dr \right| \leq T \|a \cdot \Phi'\|_{L^\infty(x_m, x_M)}$$

for all $\tau \in (0, t_0)$ and $x \in (0, x_1)$. Note that x_m does not depend here on $\tau \in (0, t_0)$, contrary to the construction from Eq. (4.13). Finally, by Lemma 2.6,

$$\left| \int_{\tau}^{\tau+\tau_0} (a \cdot \Phi')(X(r;\tau,x))\, dr \right| \leq \frac{1}{\delta} \int_0^{x_0} |\Phi'(z)|\, dz.$$

Combining these results we obtain that $\dfrac{dB(t;\tau,A(x))}{d\tau}$ is uniformly bounded in $\tau \in (0, t)$ and $x \in (0, x_1)$, namely

$$\left| \frac{d}{d\tau} B(t;\tau,A(x)) \right| \leq (\rho + \|\Phi\|_{L^\infty(0,x_M)}) \exp\left(\frac{1}{\delta} \|\Phi'\|_{L^1(0,x_0)} + T\|a\Phi'\|_{L^\infty(x_m,x_M)} \right).$$

Step 3. We now pass to the limit $x \to 0$ in Eq. (4.8) where the interchangement of limits and integrals is justified and we obtain

$$\sigma_t^{-1}(s) = A^{-1}\left(\int_s^t (u(\tau) - \Phi_0) \exp\left(- \int_\tau^t (a \cdot \Phi')(\sigma_r^{-1}(\tau))\, dr \right) d\tau \right)$$

for all $t \in (0, T)$ and $s \in (0, t)$. Clearly the right hand side is continuously diferentiable since A^{-1} is and we easily identify the derivative of $\sigma_t^{-1}(s)$.

□

Acknowledgments

J. C. acknowledges support from MICINN, projects MTM2017-91054-EXP and RTI2018-098850-B-IOO; he also acknowledges support from Plan Propio de Investigación, Universidad de Granada, Programa 9 -partially through FEDER (ERDF) funds-. E. H. acknowledges support from FONDECYT Iniciación n° 11170655. R. Y. do not have to thank the French National Research Agency for its financial support but he kindly thanks it for the excellent reviews embellished with arguments based on scientific and cultural novelties in the expertise of his yearly application file during the last four years.

Part of this work was done while J. C. and R. Y. were visiting the Departamento de Matemática at Universidad del Bío-Bío and while E. H. and J. C. were visiting Institut Denis Poisson at Université de Tours and INRA Tours.

References

[1] D. V. Alexandrov and A. P. Malygin. Transient nucleation kinetics of crystal growth at the intermediate stage of bulk phase transitions. *J. Phys. A*, 46(45):455101, 16, 2013.

[2] D. V. Alexandrov and A. P. Malygin. Nonlinear dynamics of polydisperse assemblages of particles evolving in metastable media. *Eur. Phys. J. Spec. Top.*, 229:383–404, 2020.

[3] A. B. Burd and G. A. Jackson. Particle aggregation. *Annu. Rev. Mar. Sci.*, 1:65–90, 2009.

[4] J. Calvo, M. Doumic, and B. Perthame. Long-time asymptotics for polymerization models. *Comm. Math. Phys.*, 363(1):111–137, 2018.

[5] J. A. Carrillo and T. Goudon. A numerical study on large-time asymptotics of the Lifshitz-Slyozov system. *J. Sci. Comput.*, 20(1):69–113, 2004.

[6] J.-F. Collet and T. Goudon. On solutions of the Lifshitz-Slyozov model. *Nonlinearity*, 13(4):1239–1262, 2000.

[7] J.-F. Collet, T. Goudon, F. Poupaud, and A. Vasseur. The Beker-Döring system and its Lifshitz-Slyozov limit. *SIAM J. Appl. Math.*, 62(5):1488–1500, 2002.

[8] J.-F. Collet, T. Goudon, and A. Vasseur. Some remarks on large-time asymptotic of the Lifshitz-Slyozov equations. *J. Statist. Phys.*, 108(1-2):341–359, 2002.

[9] J. G. Conlon. On a diffusive version of the Lifschitz-Slyozov-Wagner equation. *J. Nonlinear Sci.*, 20(4):463–521, 2010.

[10] G. Crippa. Lagrangian flows and the one-dimensional Peano phenomenon for ODEs. *J. Differential Equations*, 250(7):3135–3149, 2011.

[11] J. Deschamps, E. Hingant, and R. Yvinec. Quasi steady state approximation of the small clusters in Becker-Döring equations leads to boundary conditions in the Lifshitz-Slyozov limit. *Commun. Math. Sci.*, 15(5):1353–1384, 2017.

[12] B. Desjardins. A few remarks on ordinary differential equations. *Comm. Partial Differential Equations*, 21(11-12):1667–1703, 1996.

[13] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[14] M. Doumic, T. Goudon, and T. Lepoutre. Scaling limit of a discrete prion dynamics model. *Commun. Math. Sci.*, 7(4):839–865, 2009.

[15] T. Goudon, F. Lagoutière, and L. M. Tine. Simulations of the Lifshitz-Slyozov equations: the role of coagulation terms in the asymptotic behavior. *Math. Models Methods Appl. Sci.*, 23(7):1177–1215, 2013.

[16] T. Goudon and L. Monasse. Fokker-Planck approach of Ostwald Ripening: simulation of a modified Lifshitz-Slyozov-Wagner system with a diffusive correction. *SIAM J. Sci. Comput.*, 42(1):B157–B184, 2020.

[17] M. L. Greer, L. Pujo-Menjouet, and G. F. Webb. A mathematical analysis of the dynamics of prion proliferation. *J. Theoret. Biol.*, 242(3):598–606, 2006.

[18] S. Hariz and J. F. Collet. A modified version of the Lifshitz-Slyozov model. *Appl. Math. Lett.*, 12(1):81–85, 1999.

[19] E. Hingant and R. Yvinec. Deterministic and stochastic Becker-Döring equations: past and recent mathematical developments. In *Stochastic processes, multiscale modeling, and numerical methods for computational cellular biology*, pages 175–204. Springer, Cham, 2017.

[20] G. A. Jackson and A. B. Burd. Aggregation in the marine environment. *Environ. Sci. Technol.*, 32(19):2805–2814, 1998.

[21] P. Laurençot. Weak solutions to the Lifshitz-Slyozov-Wagner equation. *Indiana Univ. Math. J.*, 50(3):1319–1346, 2001.

[22] P. Laurençot. The Lifshitz-Slyozov-Wagner equation with conserved total volume. *SIAM J. Math. Anal.*, 34(2):257–272, 2002.

[23] P. Laurençot and S. Mischler. From the Becker-Döring to the Lifshitz-Slyozov-Wagner equations. *J. Statist. Phys.*, 106(5-6):957–991, 2002.

[24] P. Laurençot and C. Walker. Well-posedness for a model of prion proliferation dynamics. *J. Evol. Equ.*, 7(2):241–264, 2007.

[25] E. Leis and C. Walker. Existence of global classical and weak solutions to a prion equation with polymer joining. *J. Evol. Equ.*, 17(4):1227–1258, 2017.

[26] I. M. Lifshitz and V. V. Slyozov. The kinetics of precipitation from supersaturated solid solutions. *J. Phys. Chem. Solids*, 19(1-2):35–50, 1961.

[27] E. V. Makoveeva and D. V. Alexandrov. Mathematical simulation of the crystal nucleation and growth at the intermediate stage of a phase transition. *Russ. Metall.*, 2018:707–715, 2018.

[28] B. Niethammer. On the evolution of large clusters in the Becker-Döring model. *J. Nonlinear Sci.*, 13(1):115–155, 2003.

[29] B. Niethammer and R. L. Pego. On the initial-value problem in the Lifshitz-Slyozov-Wagner theory of Ostwald ripening. *SIAM J. Math. Anal.*, 31(3):467–485, 2000.

[30] B. Niethammer and R. L. Pego. Well-posedness for measure transport in a family of nonlocal domain coarsening models. *Indiana Univ. Math. J.*, 54(2):499–530, 2005.

[31] S. Prigent, A. Ballesta, F. Charles, N. Lenuzza, P. Gabriel, L. M. Tine, H. Rezaei, and D. M. An efficient kinetic model for assemblies of amyloid fibrils and its application to polyglutamine aggregation. *PLoS One*, 7(11):e43273, 2012.

[32] A. Schlichting. Macroscopic limit of the Becker-Döring equation *via* gradient flows. *ESAIM Control Optim. Calc. Var.*, 25:Art. 22, 36, 2019.

[33] V. A. Shneidman. Transient nucleation with a monotonically changing barrier. *Phys. Rev. E*, 82(3):031603, 2010.

[34] V. A. Shneidman. Time-dependent distributions in self-quenching nucleation. *Phys. Rev. E*, 84(3):031602, 2011.

[35] G. Simonett and C. Walker. On the solvability of a mathematical model for prion proliferation. *J. Math. Anal. Appl.*, 324(1):580–603, 2006.

[36] J. Sotomayor. *Lições de equações diferenciais ordinárias*, volume 11 of *Projeto Euclides [Euclid Project]*. Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1979.

[37] G. Stoltz and P. Terrier A mathematical justification of the finite time approximation of Becker-Dring equations by a Fokker-Planck dynamics. *arXiv:1810.01462*, 2019

[38] J. J. L. Velázquez. The Becker-Döring equations and the Lifshitz-Slyozov theory of coarsening. *J. Statist. Phys.*, 92(1-2):195–236, 1998.

[39] O. Wurl, L. Wurl, L. Miller, K. Johnson, and S. Vagle. Formation and global distribution of sea-surface microlayers. *Biogeosciences*, 8(1):121–135, 2011.