Nucleation Time in Coagulation-Fragmentation models

Romain Yvinec[1], **Julien Deschamps**[2] and **Erwan Hingant**[3]

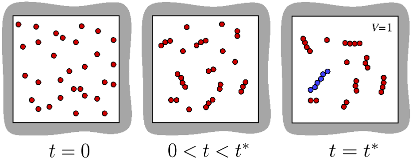
[1]BIOS group INRA Tours, France.

[2]DIMA, Universita degli Studi di Genova, Italy.

[3]CI²MA, Universidad de Concepción, Chile

Becker-Döring model for Nucleation

Reversible one-step agregation



$$C_i + C_1 \underset{b_{i+1}}{\overset{a_i}{\rightleftharpoons}} C_{i+1} \qquad (1)$$

où $C_i = \sharp\{\text{agregates of size i}\}$.

The **nucleation time** is given by the following waiting time,

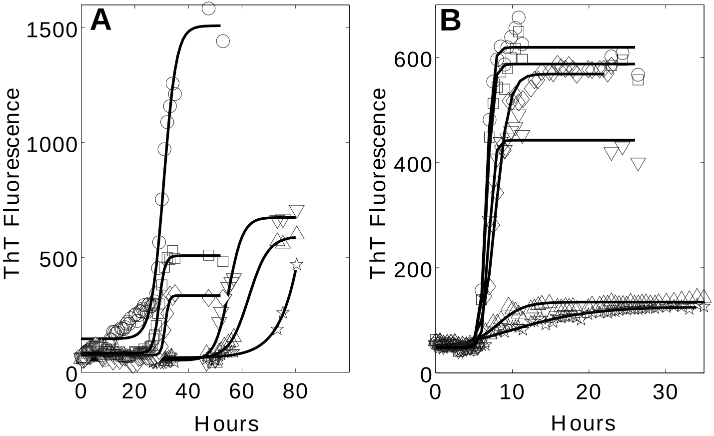
$$t^* = \inf\{t \geqslant 0 : C_N(t) = 1 \mid (C_i(0))_{i\geqslant 1}\}, \qquad (2)$$

Typical initial condition $C_i(t=0) = M\delta_{i=1}$.
N is a size of the nucleus : it's a parameter of the model (with M,a_i, b_i).

Remark

$$C_1(t) + \sum_i iC_i(t) \equiv M\,.$$

This model is used to understand spontaneous protein poylmerization experiments



“Large number limit” of the nucleation time

What are the dependencies of the nucleation time with respect to the model parameters ?
Analytical approximation and numerical simulations showed (R.Y., Maria R. D’Orsogna, Tom Chou, J. Chem. Phys. 2012) :

- non-monotone behavior with respect to the detachment rate b
- discrete size effect due to specific configuration that are “traps”.

Here, we ask : what is the nucleation time for very large nucleus

$$\lim_{N\to\infty} t^* \tag{3}$$

General idea : rescaling strategy

Consider the deterministic irreversible aggregation model

$$
\begin{aligned}
\frac{dc_i}{dt} &= a_i c_1 (c_{i-1} - c_i), \\
c_1 + \sum_i i c_i &= m, \\
t^* &= \inf\{t \geqslant 0 : c_N(t) = \rho m \mid c_1(0) = m\}
\end{aligned}
$$

Then with $c_i' = \frac{c_i}{m}$, $\tau = mt$,

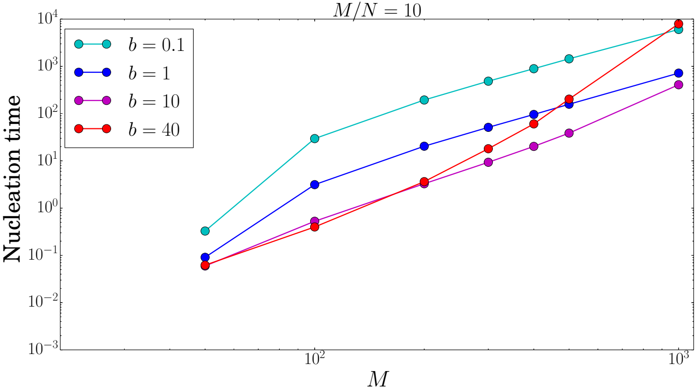
$$
\begin{aligned}
\frac{dc_i'}{d\tau} &= a_i c_1' (c_{i-1}' - c_i'), \\
c_1' + \sum_i i c_i' &= 1, \\
\tau^* &= \inf\{t \geqslant 0 : c_N'(t) = \rho \mid c_1'(0) = 1\}
\end{aligned}
$$

This strategy shows that

$$
t^* = \frac{C^{te}}{m}.
$$

General idea : rescaling strategy

Numerical simulation can give hint on the “good” rescaling choice



Choose a fix state space

With $\varepsilon = \dfrac{1}{N} \to 0$, we obtain a fix state space by studying (for appropriately rescaled C_i^ε)

$$\mu_t^\varepsilon(.) = \sum_{i \geqslant 2} \varepsilon^\alpha C_i^\varepsilon(\varepsilon^\gamma t)\delta_{i\varepsilon}(.) \in \mathcal{M}_b(\mathbb{R}^+) \tag{4}$$

The nucleation time is thus

$$t_\varepsilon^* = \inf\{t \geqslant 0 : \mu_t^\varepsilon(\{1\}) > 0 \mid \mu_0^\varepsilon\}, \tag{5}$$

Rescaled Equation (1)

$$
\begin{aligned}
\sum_i \varphi(i\varepsilon) C_i^\varepsilon(t) &= \sum_i \varphi(i\varepsilon) C_i^\varepsilon(0) + \mathcal{O}_t^{\varepsilon,\varphi} \\
&+ \int_0^t \varphi(2\varepsilon) \left[a_1^\varepsilon (C_1^\varepsilon(s))^2 - b_2^\varepsilon C_2^\varepsilon(s) \right] ds \\
&+ \int_0^t \sum_i (\varphi(i\varepsilon + \varepsilon) - \varphi(i\varepsilon)) a_i^\varepsilon C_1^\varepsilon(s) C_i^\varepsilon(s)\, ds \\
&+ \int_0^t \sum_i (\varphi(i\varepsilon - \varepsilon) - \varphi(i\varepsilon)) b_i^\varepsilon C_i^\varepsilon(s)\,.
\end{aligned}
$$

$$
\begin{aligned}
\langle \mu_t^\varepsilon, \varphi \rangle = \langle \mu_{\mathrm{in}}^\varepsilon, \varphi \rangle + \mathcal{O}_t^{\varepsilon,\varphi} \\
\int_0^t \varphi(2\varepsilon) \left[a_1^\varepsilon (C_1^\varepsilon(s))^2 - b_2^\varepsilon C_2^\varepsilon(s) \right] ds \\
+ \varepsilon \int_0^t \int \Delta_\varepsilon(\varphi)(a^\varepsilon(x) C_1^\varepsilon(s) - b^\varepsilon(x)) \mu_s^\varepsilon(dx)\, ds \quad (6)
\end{aligned}
$$

Rescaled Equation (2)

Under appropriate scaling (flux of order ε^{-1}), and by compactness argument, everything converge nicely except the red term !

$$\langle \mu_t^\varepsilon, \varphi \rangle = \langle \mu_{\rm in}^\varepsilon, \varphi \rangle + \mathcal{O}_t^{\varepsilon,\varphi}$$
$$\int_0^t \varphi(2\varepsilon) \left[a_1^\varepsilon (C_1^\varepsilon(s))^2 - b_2^\varepsilon C_2^\varepsilon(s) \right] ds$$
$$+ \int_0^t \int_0^{+\infty} \Delta_\varepsilon(\varphi)(a^\varepsilon(x) C_1^\varepsilon(s) - b^\varepsilon(x)) \mu_s^\varepsilon(dx)\, ds \quad (7)$$

We need to look at the term $C_2^\varepsilon = \langle \mu_s^\varepsilon, 1_{2\varepsilon} \rangle$!

Fast variable

The equations on C_2^ε involves C_3^ε, which involves C_4^ε and so on...and there are **fast variables**.

$$C_{i-1}^\varepsilon \underset{\frac{1}{\varepsilon} b^\varepsilon(\varepsilon i) C_i^\varepsilon}{\overset{\frac{1}{\varepsilon} a^\varepsilon(\varepsilon(i-1)) C_1^\varepsilon C_{i-1}^\varepsilon}{\rightleftharpoons}} C_i^\varepsilon \underset{\frac{1}{\varepsilon} b^\varepsilon(\varepsilon(i+1)) C_{i+1}^\varepsilon}{\overset{\frac{1}{\varepsilon} a^\varepsilon(\varepsilon i) C_1^\varepsilon C_i^\varepsilon}{\rightleftharpoons}} C_{i+1}^\varepsilon,$$

We cannot hope a convergence in a standard function space. We need a functional space that do not see the fast variations, such as $\mathcal{M}(\mathbb{R}^+, l_1(\mathbb{R}^+))$, (with respect to the weak topology) for the occupation measure, defined by, for measurable sets U of l^1,

$$\Gamma^\varepsilon\Big([0, T] \times U\Big) := \int_0^t \mathbf{1}_{\{(C_i^\varepsilon(s))_i \in U\}} ds$$

Theorem

(Under some conditions...) $(\mu^\varepsilon, (C_i^\varepsilon))$ *converges in* $\mathcal{D}(\mathbb{R}^+, (\mathcal{M}, (1+x)dx)) \times \mathcal{M}(\mathbb{R}^+, l_1(\mathbb{R}^+))$ *towards*

$$
\begin{aligned}
\langle \mu_t, \varphi \rangle &= \langle \mu_{\mathrm{in}}, \varphi \rangle + \int_0^t \varphi(0) \left[a_1 (C_1(s))^2 - b_2 C_2(s) \right] \\
&\quad + \int_0^t \int_0^{+\infty} \varphi'(x)(a(x) C_1(s) - b(x)) \mu_s(dx)\, ds \,. \\
\langle \mu_t, \mathrm{id} \rangle + C_1(t) &= m := \langle \mu_0, \mathrm{id} \rangle + C_1(0)
\end{aligned}
$$

And $(C_i(t))_{i \geqslant 2}$ *is a stationary solution in* l_1 *of the following deterministic Becker-Döring system (***for constant** $C_1 = C_1(t)$*)*

$$
\begin{aligned}
\dot{C}_2 = 0 &= -\Big(\overline{a}_2 C_1 C_2 - \overline{b}_3 C_3 \Big), \\
\dot{C}_i = 0 &= \Big(\overline{a}_{i-1} C_1 C_{i-1} - \overline{b}_i C_i \Big) - \Big(\overline{a}_i C_1 C_i - \overline{b}_{i+1} C_{i+1} \Big) \,.
\end{aligned}
$$

where $\overline{a}_i$, $\overline{b}_i$ *depends on the behavior of* a, b *at* 0

For $a(x) = \overline{a}x^r + o(x)$, $b(x) = \overline{b}x^r + o(x)$, $r < 1$, the above equation reduces to, for $C_1(t) > \frac{\overline{b}}{\overline{a}}$,

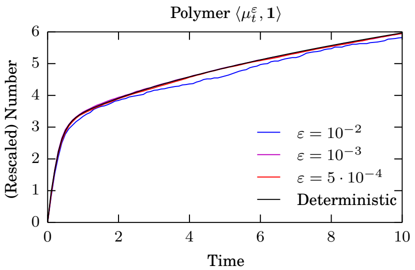
$$\begin{aligned} \langle \mu_t, \varphi \rangle &= \langle \mu_{\mathrm{in}}, \varphi \rangle + \int_0^t \varphi(0) a_1 (C_1(s))^2 \\ &\quad + \int_0^t \int_0^{+\infty} \varphi'(x)(a(x) C_1(s) - b(x)) \mu_s(dx)\, ds\,. \\ \langle \mu_t, \mathrm{id} \rangle + C_1(t) &= m := \langle \mu_0, \mathrm{id} \rangle + C_1(0) \end{aligned}$$

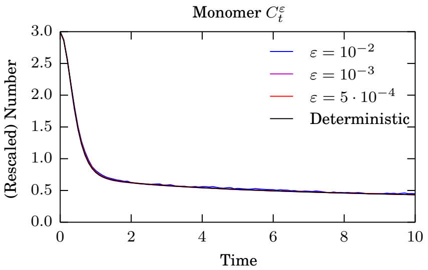
Remark

The boundary condition is : flux at 0 = dimerization rate

Numerical illustration

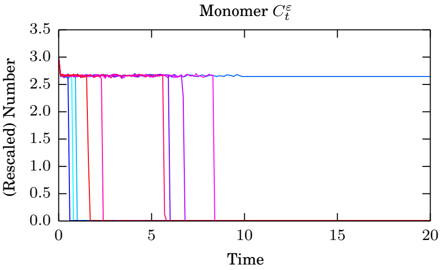
- $a(x) \equiv 1$,
  $b(x) = x$,
- Incoming charcateristics.
- Video

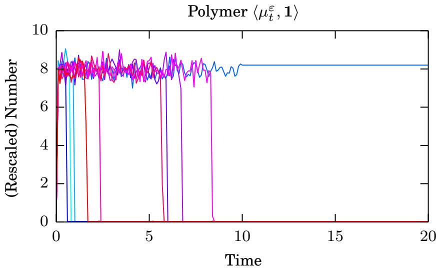




Numerical illustration and further work

- $a(x) \equiv x$,
  $b(x) = 1$,
- outgoing charcateristics.
- Video : **Metastability**





Merci !

- *First passage times in homogeneous nucleation and self-assembly*, R.Y., Maria D'Orsogna and Tom Chou (Journal of Chemical Physics (2012) 137 :244107)
- *From a stochastic Becker-Döring model to the Lifschitz-Slyozov equation with boundary value*, Julien Deschamps, Erwan Hingant and R.Y., arXiv :1412.5025 (2014)

Consider a sequence of $(\tilde{a}_i^\varepsilon)$, $(\tilde{b}_i^\varepsilon)$, $(\tilde{C}_i^\varepsilon(0))$, $(\tilde{M}^\varepsilon)$. Let $(\tilde{C}_i^\varepsilon(t))$ be the corresponding solution and define (suppose $a_1^\varepsilon, b_2^\varepsilon, a^\varepsilon, b^\varepsilon$ and $C_1^\varepsilon(0), \mu^\varepsilon(0, dx)$ converges in an appropriate sense)

$$
\begin{aligned}
a_i^\varepsilon &:= \varepsilon^A \tilde{a}_i^\varepsilon\,, \quad \forall i \geqslant 2\,,\\
b_i^\varepsilon &:= \varepsilon^B \tilde{b}_i^\varepsilon\,, \quad \forall i \geqslant 3\,,\\
a_1^\varepsilon &:= \varepsilon^{A_1} \tilde{a}_1^\varepsilon\,,\\
b_2^\varepsilon &:= \varepsilon^{B_1} \tilde{b}_2^\varepsilon\,.
\end{aligned}
\qquad
\begin{aligned}
\chi_i^\varepsilon &:= \mathbf{1}_{[(i-1/2)\varepsilon^\beta,(i+1/2)\varepsilon)}\,,\\
a^\varepsilon(x) &:= \sum_{i\geqslant 2} a_i^\varepsilon \chi_i^\varepsilon(x)\,,\\
b^\varepsilon(x) &:= \sum_{i\geqslant 3} b_i^\varepsilon \chi_i^\varepsilon(x)\,.
\end{aligned}
$$

We then define the variables

$$
\begin{aligned}
C_i^\varepsilon &= \varepsilon^\alpha \tilde{C}_i^\varepsilon\,, \forall i \geqslant 2\,,\\
C_1^\varepsilon &= \varepsilon^\theta \tilde{C}_1^\varepsilon\,.
\end{aligned}
\qquad
\begin{aligned}
\mu^\varepsilon(t, dx) &= \sum_{i\geqslant 2} C_i^\varepsilon(t)\delta_{i\varepsilon^\beta}(dx)\,,\\
M^\varepsilon &= \varepsilon^{\alpha+\beta}\tilde{M}^\varepsilon\,.
\end{aligned}
$$

The above results hold with the following choices

$$
\begin{aligned}
\theta &= \alpha+\beta\,,\\
A &= -\alpha\,,\\
B &= \beta\,,
\end{aligned}
\qquad
\begin{aligned}
A_1 &= -\alpha-2\beta\,,\\
B_1 &= 0\,,
\end{aligned}
\qquad
\begin{aligned}
a_i^\varepsilon &\sim \overline{a}_i \varepsilon^{r_a\beta},\ \forall i \geqslant 2\\
b_i^\varepsilon &\sim \overline{b}_i \varepsilon^{r_b\beta},\ \forall i \geqslant 2\,,\\
0 &\leqslant \min(r_a, r_b) < 1\,.
\end{aligned}
$$

Concrete Example

Consider a sequence of $(\tilde{a}_i^\varepsilon)$, $(\tilde{b}_i^\varepsilon)$, $(\tilde{C}_i^\varepsilon(0))$, $(\tilde{M}^\varepsilon)$. Let $(\tilde{C}_i^\varepsilon(t))$ be the corresponding solution and define (suppose $a_1^\varepsilon, b_2^\varepsilon, a^\varepsilon, b^\varepsilon$ and $C_1^\varepsilon(0), \mu^\varepsilon(0, dx)$ converges in an appropriate sense)

$$
\begin{aligned}
a^\varepsilon(x) &:= \frac{1}{\varepsilon} \sum_{i \geqslant 2} \tilde{a}_i^\varepsilon \chi_i^\varepsilon(x)\,, \\
b^\varepsilon(x) &:= \varepsilon \sum_{i \geqslant 3} \tilde{b}_i^\varepsilon \chi_i^\varepsilon(x)\,.
\end{aligned}
\qquad
\begin{aligned}
a_1^\varepsilon &:= \frac{1}{\varepsilon^3} \tilde{a}_1^\varepsilon\,, \\
b_2^\varepsilon &:= \tilde{b}_2^\varepsilon\,, \\
\tilde{a}_i^\varepsilon &\sim \overline{a}_i \varepsilon^{1+r_a}, \\
\tilde{b}_i^\varepsilon &\sim \overline{b}_i \varepsilon^{r_b - 1}, \ \min(r_a, r_b) < 1\,.
\end{aligned}
$$

We then define the variables

$$
C_1^\varepsilon = \varepsilon^2 \tilde{C}_1^\varepsilon\,, \qquad \mu^\varepsilon(t, dx) = \varepsilon \sum_{i \geqslant 2} \tilde{C}_i^\varepsilon(t) \delta_{i\varepsilon^\beta}(dx)\,.
$$